Ken Mei Test 1 Cheat Sheet

$\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln(x) = \frac{1}{x}$, $\frac{d}{dx}-\cos(x) = \sin(x)$, $\frac{d}{dx}-\sin(x) = -\cos(x)$, $\frac{d}{dx}\cot(x) = -\csc^2(x)$, $\frac{1}{\cos^2(x)} = \sec^2(x)$ | $\frac{1}{\sin^2(x)} = \csc^2(x)$, $\frac{1}{\tan^2(x)} = \cot^2(x)$

$\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, $\frac{d}{dx}\tan(x) = \sec^2(x)$, $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, $\frac{d}{dx}\csc(x) = -\cot(x)\cdot\csc(x)$

$\int\sin(x)\,dx = -\cos(x)+C$, $\int\cos(x)\,dx = \sin(x)+C$, $\int\tan(x)\,dx = -\ln|\cos(x)|+C$, $\int\sec^2(x)\,dx = \tan(x)+C$, $\int\tan(x)\sec(x)\,dx = \sec(x)+C$, $\int\csc^2(x)\,dx = -\cot(x)+C$

$\int\csc(x)\,dx = \ln|\tan(\frac{x}{2})|+C$, $\int\cot(x)\,dx = \ln|\sin(x)|+C$, $\int u\,dv = uv - \int v\,du$ (Int by Parts),

Chain Rule: $\frac{d}{dx}e^{g(x)}\cdot g'(x)$, $\sec\theta = \frac{Hyp}{Adj}$, $\cot\theta = \frac{Adj}{Opp}$, $\csc = \frac{Hyp}{Opp}$, $\cot = \frac{\cos\theta}{\sin\theta}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{1}{\cos^2(x)} = \sec^2(x)$, $\frac{1}{\sin^2(x)} = \csc^2(x)$

$\int\frac{1}{1+y^2} = \tan^{-1}(y)$

$\frac{1}{\tan^2(x)} = \cot^2(x)$

$\int\frac{1}{\cos^2(x)}dx = \tan(x)+C$

$\int\frac{1}{\sin^2(x)}dx = -\cot(x)+C$

$\int\frac{1}{\tan^2(x)}dx = -x-\cot(x)+C$

Trig Sub:

$\sqrt{a^2-x^2}$ Let $x = a\sin\theta$; $\sqrt{a^2-x^2} = a\cos\theta$

$\sqrt{a^2+x^2}$ Let $x = a\tan\theta$; $\sqrt{a^2+x^2} = a\sec\theta$

$\sqrt{x^2-a^2}$ Let $x = a\sec\theta$; $\sqrt{x^2-a^2} = \pm a\tan\theta$

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

Linear Homogeneous

$y' + p(x)y = 0$

$y = Ce^{-\int p(x)dx}$

Separable Equations:

$h(y)y' = g(x)$

one side has y and the other has x

Linear Non homogeneous

$y' + P(x)y = f(x)$ $f(x) \neq 0$

① solve associated homogeneous equation

② use variation of parameters to solve orginal equation

③ Let y_1 be solution of $y = uy_1$

$y' = u'y_1 + uy_1'$

$u = \int\frac{f(x)}{y_1}dx$

Exact Equations:

$M(x,y)dx + N(x,y)dy = 0$ is exact if $M_y = N_x$

• If $M(x,y)dx$ the $N(x,y)dy = 0$ is exact then there's an implicit solution $F(x,y) = C$, where $F_x = M$, $F_y = N$

Bernomoulli Equation: $y' + P(x)y = f(x)y^r$

Special cases: ① $r = 0$: $y' + P(x)y = f(x) \rightarrow$ Linear

② $r = 1$: $y' + P(x)y = f(x)y$

$y' + (P(x) - f(x))y = 0 \rightarrow$ linear homogenous + separable equation

To solve: ① y_1 is any solution of homogenous equation $y' + P(x)y = 0$

② Solve using variations of parameters set $y = uy_1$

$y' = u'y_1 + uy_1'$, $\frac{u'}{u^r} = f(x)y_1^{r-1}$

Nonlinear Homogeneous: ① Test whether it's homogenous or not (if it depends only on $\frac{y}{x}$), ② $y = uy_1$ (let $y_1 = x$), ③ Plug $y = uy_1 = ux$ into $y' = f(x,y)$ to find out what u must satisfy

$u = \frac{y}{x}$, $f(x,y) = g(u)$, $y = ux$, $y' = u'x + u$, $\frac{u'}{g(u)-u} = \frac{1}{x}$

Webwork-Separable $\rightarrow y' = (1-2x)y^2$, $y(0) = 1/-6$ General Solution explicitly

① $\frac{dy}{dx} = (1-2x)y^2$, ② $\frac{\frac{dy}{dx}}{y^2} = 1-2x$, ③ $\int\frac{dy}{y^2} = \int(1-2x)dx$

$\int\frac{1}{y^2}dy = \int(1-2x)dx \rightarrow -\frac{1}{y} = x - x^2 + C \rightarrow y(x) = \frac{-1}{x-x^2+C}$

$\frac{-1}{\frac{-1}{-6}} = 0 - 0^2 + C \rightarrow \frac{-1}{1}\cdot\frac{-6}{1} = 6 = C \rightarrow y(x) = \frac{-1}{x-x^2+6}$

Interval of Validity for all except

$x - x^2 + 6 = 0 \rightarrow -x^2 + x + 6 = 0 \rightarrow -(x+2)(x-3)$

$-x-2 = 0$, $x-3 = 0$; $x = -2$, $x = 3$. Interval of Validity $= (-2,3)$

Example problem from Webwork: First Order Eq'n Linear Nonhomogeneous

$y' - 9y = 2e^{9t}$, $y(0) = 3.2$

Step 1: Solve $y' + y = 0$

$y' - 9y = 0$, $+9y$ $+9y$, $y' = 9y$, $\frac{y'}{y} = 9 \rightarrow \int\frac{y'}{y}dx = \int 9dx$

$= \int\frac{1}{y}dy = \int 9dx$

$= \ln|y| = 9t + C$

$y = Ce^{9t+C} \rightarrow y_1 = e^{9t}$

Step 2: $u = \int\frac{f(t)}{y_1} = \int\frac{2e^{9t}}{e^{9t}}dt$

$u = 2t + C$; $y = u\cdot y_1 = \underbrace{(2t+C)e^{9t}}_{\text{general solution}}$

$y(0) = 3.2$

$y(0) = 2(0)e^{9(0)} + Ce^{9(0)} \rightarrow C = 3.2$

$= 2te^{9t} + 3.2e^{9t}$ → Particular Solution

Webwork Exact Equation: $8xy^2 + kx^2y + (8x^2y - 7x^3)y' = 0$

① $M = 8x^2y^2 + kx^2y$, $N = 8x^2y - 7x^3$ DE is exact if $\frac{dm}{dy} = \frac{dN}{dx}$

$\frac{dm}{dy} = 16xy + kx^2$, $\frac{dN}{dx} = 16xy - 21x^2 \rightarrow 16xy + kx^2 = 16xy - 21x^2$ $k = -21$

Solution of ① is given by $\int Mdx + \int Ndy = C$, $\int Mdx = \int(8x^2y^2 - 21x^2y)dx$

$= 4x^2y^2 - 7x^3y + C$, $\int Ndy = \int(8x^2y - 7x^3)dy = 4x^2y^2 - 7x^3y + C$, Genearal Solution →

$\rightarrow 4x^2y^2 - 7x^3 = C$

Webwork - Nonlinear Bernoulli $\rightarrow y' + 6y = 9\frac{1}{y^8}$

$P(t) = 6$, $f(t) = 9$, $r = -8$, $r-1 = -9$

① $y' + 6y = 0$, $y = Ce^{-\int P(t)dt}$

$= Ce^{-\int 6dt} \rightarrow Ce^{-6t} \rightarrow y_1 = e^{-6t}$

② $y = u\cdot y_1$

$\frac{u'}{u^r} = f(t)y_1^{r-1} \rightarrow \frac{u'}{u^{-8}} = 9(e^{-6t})^{-9}$

$\int u^8 du = 9\int e^{54t}dt$

$\frac{u^9}{9} = 9\cdot\frac{e^{54t}}{54} + C$

$\cancel{9}\cdot\frac{u^9}{\cancel{9}} = \left(\frac{1}{6}e^{54t} + C\right)\cdot 9$

$\sqrt[9]{u^9} = \sqrt[9]{\frac{9}{6}e^{54t} + C}$

$u = \left(\frac{9}{6}e^{54t} + C\right)^{\frac{1}{9}}$

$y = u\cdot y_1$ so

$y(t) = \left(\frac{9}{6}e^{54t} + C\right)^{\frac{1}{9}}\cdot(e^{-6t})$

Webwork - Nonlinear Homogeneous $\rightarrow x^2y' = xy + y^2$

$y' = \frac{xy}{x^2} + \frac{y^2}{x^2} \rightarrow y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$ Let $u = \frac{y}{x}$

$q(u) = u + u^2$, step 1: $y_1 = x$, step 2: $\frac{u'}{q(u)-u} = \frac{1}{x} \rightarrow \frac{u'}{u+u^2-u} = \frac{1}{x}$

$\frac{u'}{u^2} = \frac{1}{x} \rightarrow \int\frac{u'}{u^2} = \int\frac{1}{x} \rightarrow \int\frac{1}{u^2}du = \int\frac{1}{x} \rightarrow \int u^{-2}du = \int\frac{1}{x}$

$= -\frac{1}{u} = \ln|x| + C \rightarrow \cancel{u}\cdot\frac{-1}{\cancel{u}} = (\ln|x| + C)\cdot u \rightarrow \frac{-1}{\ln|x|+C} = \frac{\cancel{(\ln|x|+C)}\cdot u}{\cancel{\ln|x|+C}}$

$u = \frac{-1}{\ln|x|+C} \rightarrow y(x) = \frac{-1}{\ln|x|+C}$