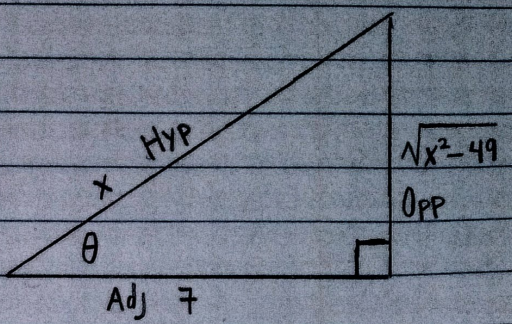
6) $\int \frac{\sqrt{x^2-49}}{x^4}\,dx$ Apply standard 3 using Trigonometric Substitution

For integrals involving $\sqrt{x^2-a^2}$, let $x = a\sec\theta$

Then $\sqrt{x^2-a^2} = \pm a\tan\theta$



1) Use Pythagorean Identity $\sec^2\theta - \tan^2\theta = 1$

$$\cancel{\sec^2\theta} - \tan^2\theta = 1$$
$$-\cancel{\sec^2\theta} \qquad -\sec^2\theta$$
$$-\tan^2\theta = 1-\sec^2\theta$$
$$-1 \qquad -1$$
$$\tan^2\theta = -1+\sec^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1$$

$\sin\theta = \frac{\sqrt{x^2-49}}{x}$

$\cos\theta = \frac{7}{x}$

$\tan\theta = \frac{\sqrt{x^2-49}}{7}$

$\sin\theta = \frac{OPP}{HYP}$

$\cos\theta = \frac{ADJ}{HYP}$

$\tan\theta = \frac{OPP}{ADJ}$

$\csc\theta = \frac{HYP}{OPP}$

$\sec\theta = \frac{HYP}{ADJ}$

$\cot\theta = \frac{ADJ}{OPP}$

2) Since $\sqrt{x^2-a^2}$ is involved where $x = a\sec\theta$, then

$x = \sqrt{x^2-a^2} = \sqrt{x^2-49} = 7\sec\theta \rightarrow dx = 7\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{(7^2\sec^2\theta)-49}}{(7\sec\theta)^4}(7\sec\theta\tan\theta\,d\theta)$$

Apply $7\sec\theta$ and replace x; the same goes for $dx = 7\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{49\sec^2\theta-49}}{7^4\sec^4\theta}(7\sec\theta\tan\theta\,d\theta) = \int \frac{\sqrt{49(\sec^2\theta-1)}}{7^4\sec^4\theta}(7\sec\theta\tan\theta\,d\theta)$$

$$= \int \frac{\sqrt{49\tan^2\theta}}{2401\sec^4\theta}(7\sec\theta\tan\theta\,d\theta) = \int \frac{7\tan\theta\, 7\sec\theta\tan\theta}{2401\sec^4\theta}\,d\theta$$

$$= \int \frac{49\sec\theta\tan^2\theta}{2401\sec^4\theta}\,d\theta = \frac{49}{2401}\int \frac{\cancel{\sec\theta}\tan^2\theta}{\sec^{\cancel{4}3}\theta}\,d\theta = \frac{1}{49}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta = \frac{1}{49}\int \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^3\theta}}\,d\theta$$

Because $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$ and $\sec^3\theta = \frac{1}{\cos^3\theta}$

$$= \frac{1}{49}\int \frac{\sin^2\theta\cos^{\cancel{3}}\theta}{\cancel{\cos^2\theta}}\,d\theta = \frac{1}{49}\int \sin^2\theta\cos\theta\,d\theta$$

3) Continue to apply the u-substitution by letting $u = \sin\theta$

$$\frac{1}{49}\int \sin^2\theta\cos\theta\,d\theta = \frac{1}{49}\int u^2\,du = \frac{1}{49}\left(\frac{u^{2+1}}{2+1}\right) = \frac{1}{49}\left(\frac{u^3}{3}\right) = \frac{u^3}{147} = \frac{\sin^3\theta}{147}$$

$$= \frac{1}{147}\left(\frac{\sqrt{x^2-49}}{x}\right)^3 = \frac{1}{147}\left(\frac{(x^2-49)^{3/2}}{x^3}\right) = \boxed{\frac{(x^2-49)^{3/2}}{147x^3} + C}$$