Sections 3.8, 5.2, 6.1 and 6.2 problems 1-9 **Exam is 1 hour 15 minutes.**

Ok: handwritten notes, calculator (TI 83/84)

Not ok: printouts, book, **TI 89,** laptop, tablet, cell phone or other handheld

1. (20 pts) Given an RLC circuit with R=2 ohms, L=1/10 henries, C=1/260 farads and E(t)=sin60t; find the
   a. differential equation for the current I (remember to use E'(t), not E(t));
   $0.1I''+2I'+260I=60\cos 60t \Rightarrow I''+20I'+2600I=600\cos 60t$
   b. the form of the transient current I_h;
   $r^2+20r+2600=0 \Rightarrow (r+10)^2+50^2=0 \Rightarrow r=-10\pm 50i$
   so $I_h=e^{-10t}\left(c_1\cos 50t+c_2\sin 50t\right)$
   c. the periodic current I_p; Form of solution is $I_p=A\cos 60t+B\sin 60t$
   Taking the derivatives we get
   $$I_p{}'=60(-A\sin 60t+B\cos 60t) \text{ and } I_p{}''=-3600(A\cos 60t+B\sin 60t)$$
   Plugging into the ODE, we get
   $$-3600(A\cos 60t+B\sin 60t)+20*60(-A\sin 60t+B\cos 60t)+2600(A\cos 60t+B\sin 60t)=600\cos 60t$$
   $$\Rightarrow -1000A+1200B=600 \ \& \ -1200A-1000B=0$$
   $$\Rightarrow -5A+6B=3 \ \& \ 6A+5B=0 \text{ so } B=18/61 \ \& \ A=-15/61$$
   $$\Rightarrow I_p=\left(-15\cos 60t+18\sin 60t\right)/61$$

2. (20 pts) Given a mass spring damper with m=1 kg, γ=2Ns/m, k=50N/m, F=20cos(ωt),
   a. find the value of ω that gives practical resonance;
   $$u''+2u'+50u=20\cos\omega t \ \& \ u_p=A\cos\omega t+B\sin\omega t$$
   Taking the derivatives we get:
   $$u_p{}'=\omega(-A\sin\omega t+B\cos\omega t) \text{ and } u_p{}''=-\omega^2(A\cos\omega t+B\sin\omega t)$$
   Substituting into the ODE, we get
   $$-\omega^2(A\cos\omega t+B\sin\omega t)+2\omega(-A\sin\omega t+B\cos\omega t)+50\left(A\cos\omega t+B\sin\omega t\right)=20\cos\omega t$$
   $$\Rightarrow \begin{cases}-A\omega^2+2B\omega+50A=20\\ -B\omega^2-2A\omega+50B=0\end{cases} \Rightarrow \begin{cases}\left(50-\omega^2\right)A+2\omega B=20\\ -2\omega A+\left(50-\omega^2\right)B=0\end{cases} \Rightarrow \begin{cases}2\omega\left(50-\omega^2\right)A+4\omega^2 B=40\omega\\ -2\omega\left(50-\omega^2\right)A+\left(50-\omega^2\right)^2 B=0\end{cases}$$
   $$\Rightarrow B=40\omega/\left(2500-96\omega^2+\omega^4\right) \text{ and } A=20\left(50-\omega^2\right)/\left(2500-96\omega^2+\omega^4\right)$$

For a function that is positive, maximizing the square of the function is equivalent to maximizing the original function. The square of the amplitude is

$$A^2+B^2=\left[20\left(50-\omega^2\right)/\left(2500-96\omega^2+\omega^4\right)\right]^2+\left[40\omega/\left(2500-96\omega^2+\omega^4\right)\right]^2$$
$$=400\left(2500-96\omega^2+\omega^4\right)\left(2500-96\omega^2+\omega^4\right)^{-2}=400/\left(2500-96\omega^2+\omega^4\right)$$

To maximize a fraction, it is the same as minimizing the reciprocal. Taking the derivative of $2500-96\omega^2+\omega^4$

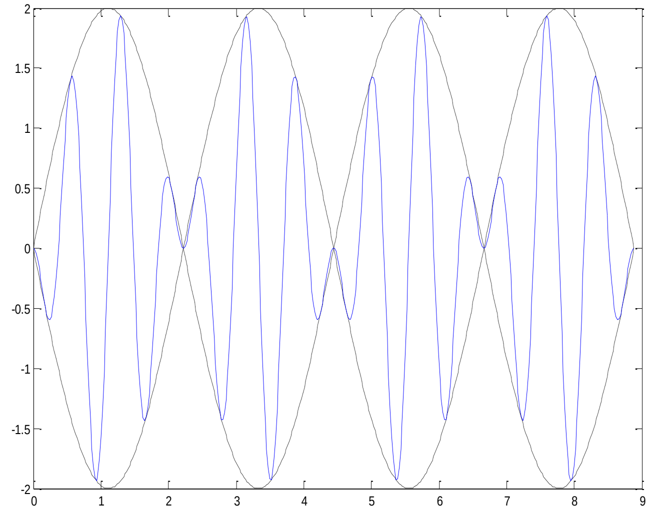
We get $4\omega^3-192\omega=4\omega\left(\omega-4\sqrt{3}\right)\left(\omega+4\sqrt{3}\right)$ so the polynomial has a min at $\omega=4\sqrt{3}\approx 6.93$

Note that the natural frequency (without damper) is $\omega_0=5\sqrt{2}\approx 7.07$ As a check, since $\dfrac{\gamma^2}{mk}=\dfrac{4}{50}$ is quite small (<.1), the resonant and natural frequencies should be close to one another, which they are.

b. if the damper is removed, then the circular frequency without a forcing function is $5\sqrt{2}$. Suppose that the forcing function has circular frequency $7\sqrt{2}$ and that the initial conditions give a solution

$$y(t) = \cos 5\sqrt{2}t - \cos 7\sqrt{2}t$$

Use the appropriate sum or difference identities to write the solution as a product of 2 trigonometric functions of different frequencies. Sketch a graph over one long period to illustrate the phenomenon of beats.



$$A - B = 5\sqrt{2}t, A + B = 7\sqrt{2}t \Rightarrow A = 6\sqrt{2}t, B = \sqrt{2}t$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\cos(7\sqrt{2}t) = \cos 6\sqrt{2}t \cos\sqrt{2}t - \sin 6\sqrt{2}t \sin\sqrt{2}t$$
$$\cos(5\sqrt{2}t) = \cos 6\sqrt{2}t \cos\sqrt{2}t + \sin 6\sqrt{2}t \sin\sqrt{2}t$$

Subtracting, we get:

$$\cos(7\sqrt{2}t) - \cos(5\sqrt{2}t) = -2\sin 6\sqrt{2}t \sin\sqrt{2}t$$

3. (20 pts) For $y'' - xy' - y = 0, \quad x_0 = 0$

a. Find the recurrence relation of a power series solution. Substituting the form of the solution and its derivatives into the equation we get:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Shift the index of the 1st series by 2 & bring coefficient x inside 2nd series and have it start at 0

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Extract the coefficients of x^n in each series to get recurrence relation:

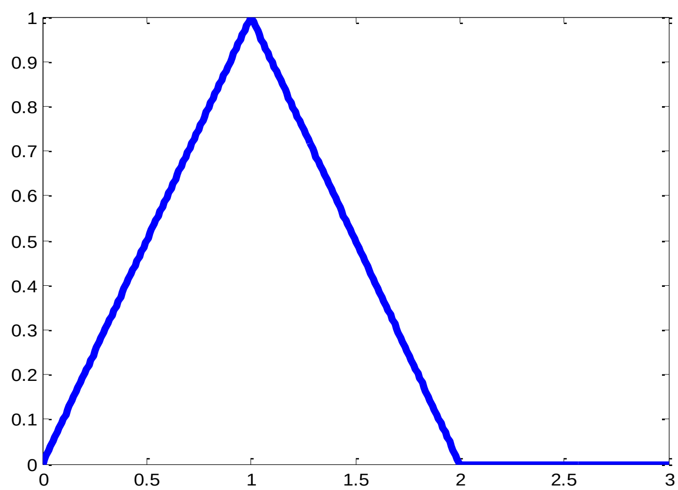
$$(n+2)(n+1) a_{n+2} - n a_n - a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{(n+1) a_n}{(n+2)(n+1)} = \frac{a_n}{(n+2)}$$

b. Use the relation to find the first 3 terms of one of the 2 basic solutions.
We find 3 terms of basic solution with even exponents (set a_1=0). We apply recursion twice:

$$a_2 = \frac{a_0}{(0+2)} = \frac{a_0}{2}; \quad a_4 = \frac{a_2}{(2+2)} = \frac{a_2}{4} = \frac{a_0}{8} \quad \Rightarrow \quad y_1(x) = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + ...\right)$$

4. (20 pts) Given $f(t) = \begin{cases} t, & [0,1) \\ 2-t, & [1,2) \\ 0, & [2,\infty) \end{cases}$

a. Graph $f(t)$ over [0,3]



b. classify as continuous, piecewise continuous, or neither

c. use the definition to find the Laplace Transform

$$F(s)=\int_0^1 te^{-st}dt+\int_1^2 (2-t)e^{-st}dt \qquad \begin{matrix} u=t & dv=e^{-st}dt \\ du=dt & v=-e^{-st}/s \end{matrix}$$

$$=\left(-\frac{te^{-st}}{s}\bigg|_0^1+\frac{1}{s}\int_0^1 e^{-st}dt\right)+2\int_1^2 e^{-st}dt-\int_1^2 te^{-st}dt$$

$$=\left(\frac{-e^{-s}}{s}-\frac{e^{-st}}{s^2}\bigg|_0^1\right)-\frac{2e^{-st}}{s}\bigg|_1^2-\left(-\frac{te^{-st}}{s}\bigg|_1^2+\frac{1}{s}\int_1^2 e^{-st}dt\right)$$

$$=\left(\frac{-e^{-s}}{s}+\frac{1-e^{-s}}{s^2}\right)+\frac{2e^{-s}-2e^{-2s}}{s}+\left(\frac{2e^{-2s}-e^{-s}}{s}+\frac{e^{-st}}{s^2}\bigg|_1^2\right)$$

$$=\left(\frac{-e^{-s}}{s}+\frac{2e^{-s}-2e^{-2s}}{s}+\frac{2e^{-2s}-e^{-s}}{s}\right)+\left(\frac{1-e^{-s}}{s^2}+\frac{e^{-2s}-e^{-s}}{s^2}\right)$$

$$=\frac{1-2e^{-s}+e^{-2s}}{s^2}$$

Some of you may be wonder whether there is an easier way to find the transform. As we work more with piecewise functions, in particular step functions, we will pick up additional tools that may enable us to find the answer with much less sweat than using the definition.

5. (20 pts) Use the table of Laplace Transforms given in class to find the inverse Laplace Transform of

$$Y(s)=\frac{5s^2+3s+30}{s^3+2s^2+10s}=\frac{5s^2+3s+30}{s(s^2+2s+1+9)}=\frac{5s^2+3s+30}{s\left((s+1)^2+3^2\right)}$$

$$=A\frac{1}{s}+B\frac{s+1}{(s+1)^2+3^2}+C\frac{3}{(s+1)^2+3^2}$$

Clearing the denominators, we get

$$A\left((s+1)^2+3^2\right)+Bs(s+1)+3Cs=5s^2+3s+30$$

Setting s=0, we get 10A=30 or A=3.
Setting s= −1, we get 9A−3C =27−3C=32 or C=−5/3
Looking at the coefficients of s, we get 2A+B+3C=6+B−5=3 or B=3
So

$$Y(s)=3\frac{1}{s}+3\frac{s+1}{(s+1)^2+3^2}+-5/3\frac{3}{(s+1)^2+3^2}$$

And $y(t)=3+3e^{-t}\cos 3t+-(5/3)e^{-t}\sin 3t$