Exam I part V guide Boyce/10ed/Ch1&2 Halleck MAT 2680 NYCCT (CUNY) Fall 2013

For part V of exam I, you will be given 3 first order equations with analytic solutions and asked to identify the type, provide the first few steps of the solution and check the solution (it will be given). Here is what I mean by the "first few steps":

1. If the equation is separable, separate the variables and express as integrals. Note what integration technique you would use to evaluate the integrals (integration by parts, partial fractions, etc.).
2. If the equation is linear, put in standard form and find and apply the IF μ. Express *y* as an integral. Note what integration technique to use to evaluate the integrals (integration by parts, partial fractions, etc.).
3. Homogeneous: perform the substitution and separate the variables.
4. Bernoulli: perform the substitution and put resulting linear equation into standard form.
5. Exact: show exactness, integrate M with respect to *x* to find Ψ up to "constant" C(*y*)

Here are the solutions to p. 133 #1-32 of the text:

1. The equation is *linear*. It can be written in the form $y' + 2y/x = x^2$, and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ yields $x^2y' + 2yx = (yx^2)' = x^4$. Integration with respect to x and division by x^2 gives that $y = x^3/5 + c/x^2$.

2. The equation is *separable*. Separating the variables gives the differential equation $(2 - \sin y)dy = (1 + \cos x)dx$, and after integration we obtain that the solution is $2y + \cos y - x - \sin x = c$.

3. The equation is *exact*. Simplification gives $(2x + y)dx + (x - 3 - 3y^2)dy = 0$. We can check that $M_y = 1 = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2 + xy + g(y)$, then $\psi_y = x + g'(y) = x - 3 - 3y^2$,

which means that $g'(y) = -3 - 3y^2$, so integrating with respect to y we obtain that $g(y) = -3y - y^3$. Therefore the solution is defined implicitly as $x^2 + xy - 3y - y^3 = c$. The initial condition $y(0) = 0$ implies that $c = 0$, so we conclude that the solution is $x^2 + xy - 3y - y^3 = 0$.

4. The equation is *linear*. It can be written as $y' + (2x - 1)y = -3(2x - 1)$, and the integrating factor is e^{x^2-x}. Multiplication by this integrating factor and the subsequent integration gives the solution $ye^{x^2-x} = -3e^{x^2-x} + c$, which means that $y = -3 + ce^{x-x^2}$. (The equation is also *separable*.)

5. The equation is *exact*. Algebraic manipulations give the symmetric form of the equation, $(2xy + y^2 + 1)dx + (x^2 + 2xy)dy = 0$. We can check that $M_y = 2x + 2y = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2y + xy^2 + x + g(y)$, then $\psi_y = x^2 + 2xy + g'(y) = x^2 + 2xy$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^2y + xy^2 + x = c$.

6. The equation is *linear*. It can be written in the form $y' + (1 + (1/x))y = 1/x$ and the integrating factor is $\mu(x) = e^{\int 1+(1/x)\,dx} = e^{x+\ln x} = xe^x$. Multiplication by $\mu(x)$ yields $xe^xy' + (xe^x + e^x)y = (xe^xy)' = e^x$. Integration with respect to x and division by xe^x shows that the general solution of the equation is $y = 1/x + c/(xe^x)$. The initial condition implies that $0 = 1 + c/e$, which means that $c = -e$ and the solution is $y = 1/x - e/(xe^x) = x^{-1}(1 - e^{1-x})$.

7. The equation is *separable*. Separation of variables gives the differential equation $y(2 + 3y)dy = (4x^3 + 1)dx$, and then after integration we obtain that the solution is $x^4 + x - y^2 - y^3 = c$.

8. The equation is *linear*. It can be written in the form $y' + 2y/x = \sin x/x^2$ and the integrating factor is $\mu(x) = e^{\int (2/x)\,dx} = e^{2\ln x} = x^2$. Multiplication by $\mu(x)$ gives $x^2y' + 2xy = (x^2y)' = \sin x$, and after integration with respect to x and division by x^2 we obtain the general solution $y = (c - \cos x)/x^2$. The initial condition implies that $c = 4 + \cos 2$ and the solution becomes $y = (4 + \cos 2 - \cos x)/x^2$.

9. The equation is *exact*. Simplification gives $(2xy + 1)dx + (x^2 + 2y)dy = 0$. We can check that $M_y = 2x = N_x$, so the equation is really exact. Integrating M with respect to x gives that $\psi(x, y) = x^2y + x + g(y)$, then $\psi_y = x^2 + g'(y) = x^2 + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the solution is defined implicitly as $x^2y + x + y^2 = c$.

10. The equation is *separable*. Factoring the terms we obtain the differential equation $(x^2 + x - 1)ydx + x^2(y - 2)dy = 0$. We separate the variables by dividing this equation by yx^2 and obtain

$$(1 + \frac{1}{x} - \frac{1}{x^2})dx + (1 - \frac{2}{y})dy = 0.$$

Integration gives us the solution $x + \ln|x| + 1/x - 2\ln|y| + y = c$. We also have the solution $y = 0$ which we lost when we divided by y.

11. The equation is *exact*. It is easy to check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x, y) = x^3/3 + xy + g(y)$, then $\psi_y = x + g'(y) =$

$x + e^y$, which means that $g'(y) = e^y$, so we obtain that $g(y) = e^y$. Therefore the solution is defined implicitly as $x^3/3 + xy + e^y = c$.

12. The equation is *linear*. The integrating factor is $\mu(x) = e^{\int dx} = e^x$, which turns the equation into $e^xy' + e^xy = (e^xy)' = e^x/(1 + e^x)$. We can integrate the right hand side by substituting $u = 1 + e^x$, this gives us the solution $ye^x = \ln(1 + e^x) + c$, i.e. $y = ce^{-x} + e^{-x}\ln(1 + e^x)$.

13. The equation is *separable*. Factoring the right hand side leads to the equation $y' = (1 + y^2)(1 + 2x)$. We separate the variables to obtain $dy/(1 + y^2) = (1 + 2x)dx$, then integration gives us $\arctan y = x + x^2 + c$. The solution is $y = \tan(x + x^2 + c)$.

14. The equation is *exact*. We can check that $M_y = 1 = N_x$. Integrating M with respect to x gives that $\psi(x, y) = x^2/2 + xy + g(y)$, then $\psi_y = x + g'(y) = x + 2y$, which means that $g'(y) = 2y$, so we obtain that $g(y) = y^2$. Therefore the general solution is defined implicitly as $x^2/2 + xy + y^2 = c$. The initial condition gives us $c = 17$, so the solution is $x^2 + 2xy + 2y^2 = 34$.

15. The equation is *separable*. Separation of variables leads us to the equation

$$\frac{dy}{y} = \frac{1 - e^x}{1 + e^x}dx.$$

Note that $1 + e^x - 2e^x = 1 - e^x$. We obtain that

$$\ln|y| = \int \frac{1 - e^x}{1 + e^x}dx = \int 1 - \frac{2e^x}{1 + e^x}dx = x - 2\ln(1 + e^x) + \tilde{c}.$$

This means that $y = ce^x(1 + e^x)^{-2}$, which also can be written as $y = c/\cosh^2(x/2)$ after some algebraic manipulations.

16. The equation is *exact*. The symmetric form is $(-e^{-x}\cos y + e^{2y}\cos x)dx + (-e^{-x}\sin y + 2e^{2y}\sin x)dy = 0$. We can check that $M_y = e^{-x}\sin y + 2e^{2y}\cos x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = e^{-x}\cos y + e^{2y}\sin x + g(y)$, then $\psi_y = -e^{-x}\sin y + 2e^{2y}\sin x + g'(y) = -e^{-x}\sin y + 2e^{2y}\sin x$, so we get that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $e^{-x}\cos y + e^{2y}\sin x = c$.

17. The equation is *linear*. The integrating factor is $\mu(x) = e^{-\int 3\,dx} = e^{-3x}$, which turns the equation into $e^{-3x}y' - 3e^{-3x}y = (e^{-3x}y)' = e^{-x}$. We integrate with respect to x to obtain $e^{-3x}y = -e^{-x} + c$, and the solution is $y = ce^{3x} - e^{2x}$ after multiplication by e^{3x}.

18. The equation is *linear*. The integrating factor is $\mu(x) = e^{\int 2\,dx} = e^{2x}$, which gives us $e^{2x}y' + 2e^{2x}y = (e^{2x}y)' = e^{-x^2}$. The antiderivative of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 0 to x. We obtain that the left hand side turns into

$$\int_0^x (e^{2s}y(s))'ds = e^{2x}y(x) - e^0 y(0) = e^{2x}y - 3.$$

The right hand side gives us $\int_0^x e^{-s^2}\,ds$. So we found that

$$y = e^{-2x}\int_0^x e^{-s^2}\,ds + 3e^{-2x}.$$

19. The equation is *exact*. Algebraic manipulations give us the symmetric form $(y^3 + 2y - 3x^2)dx + (2x + 3xy^2)dy = 0$. We can check that $M_y = 3y^2 + 2 = N_x$. Integrating M with respect to x gives that $\psi(x,y) = xy^3 + 2xy - x^3 + g(y)$, then $\psi_y = 3xy^2 + 2x + g'(y) = 2x + 3xy^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is $xy^3 + 2xy - x^3 = c$.

20. The equation is *separable*, because $y' = e^{x+y} = e^x e^y$. Separation of variables yields the equation $e^{-y}dy = e^x dx$, which turns into $-e^{-y} = e^x + c$ after integration and we obtain the implicitly defined solution $e^x + e^{-y} = c$.

21. The equation is *exact*. Algebraic manipulations give us the symmetric form $(2y^2 + 6xy - 4)dx + (3x^2 + 4xy + 3y^2)dy = 0$. We can check that $M_y = 4y + 6x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = 2y^2x + 3x^2y - 4x + g(y)$, then $\psi_y = 4yx + 3x^2 + g'(y) = 3x^2 + 4xy + 3y^2$, which means that $g'(y) = 3y^2$, so we obtain that $g(y) = y^3$. Therefore the solution is $2xy^2 + 3x^2y - 4x + y^3 = c$.

22. The equation is *separable*. Separation of variables turns the equation into $(y^2+1)dy = (x^2-1)dx$, which, after integration, gives $y^3/3 + y = x^3/3 - x + c$. The initial condition yields $c = 2/3$, and the solution is $y^3 + 3y - x^3 + 3x = 2$.

23. The equation is *linear*. Division by t gives $y' + (1 + (1/t))y = e^{2t}/t$, so the integrating factor is $\mu(t) = e^{\int(1+(1/t))dt} = e^{t+\ln t} = te^t$. The equation turns into $te^ty' + (te^t + e^t)y = (te^ty)' = e^{3t}$. Integration therefore leads to $te^ty = e^{3t}/3 + c$ and the solution is $y = e^{2t}/(3t) + ce^{-t}/t$.

24. The equation is *exact*. We can check that $M_y = 2\cos y \sin x \cos x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = \sin y \sin^2 x + g(y)$, then $\psi_y = \cos y \sin^2 x + g'(y) = \cos y \sin^2 x$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $\sin y \sin^2 x = c$.

25. The equation is *exact*. We can check that

$$M_y = -\frac{2x}{y^2} - \frac{x^2 - y^2}{(x^2+y^2)^2} = N_x.$$

Integrating M with respect to x gives that $\psi(x,y) = x^2/y + \arctan(y/x) + g(y)$, then $\psi_y = -x^2/y^2 + x/(x^2+y^2) + g'(y) = x/(x^2+y^2) - x^2/y^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^2/y + \arctan(y/x) = c$.

27. The equation can be made *exact* with an appropriate integrating factor. Algebraic manipulations give us the symmetric form $xdx - (x^2y + y^3)dy = 0$. We can check that $(M_y - N_x)/M = 2xy/x = 2y$ depends only on y, which means we will be able to find an integrating factor in the form $\mu(y)$. This integrating factor is $\mu(y) = e^{-\int 2ydy} = e^{-y^2}$. The equation after multiplication becomes

$$e^{-y^2}xdx - e^{-y^2}(x^2y + y^3)dy = 0.$$

This equation is exact now, as we can check that $M_y = -2ye^{-y^2}x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = e^{-y^2}x^2/2 + g(y)$, then $\psi_y = -e^{-y^2}x^2y + g'(y) = -e^{-y^2}(x^2y + y^3)$, which means that $g'(y) = -y^3e^{-y^2}$. We can integrate this expression by substituting $u = -y^2$, $du = -2ydy$. Integrating by parts, we obtain that

$$g(y) = -\int y^3e^{-y^2}dy = -\int \frac{1}{2}ue^u du = -\frac{1}{2}(ue^u - e^u) + c =$$

$$-\frac{1}{2}(-y^2e^{-y^2} - e^{-y^2}) + c.$$

Therefore the solution is defined implicitly as $e^{-y^2}x^2/2 - \frac{1}{2}(-y^2e^{-y^2} - e^{-y^2}) = c$,

28. The equation can be made *exact* by choosing an appropriate integrating factor. We can check that $(M_y - N_x)/N = (2-1)/x = 1/x$ depends only on x, so $\mu(x) = e^{\int(1/x)dx} = e^{\ln x} = x$ is an integrating factor. After multiplication, the equation becomes $(2yx + 3x^2)dx + x^2dy = 0$. This equation is exact now, because $M_y = 2x = N_x$. Integrating M with respect to x gives that $\psi(x,y) = yx^2 + x^3 + g(y)$, then $\psi_y = x^2 + g'(y) = x^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $x^3 + x^2y = c$.

29. The equation is *homogeneous*. (See Section 2.2, Problem 30) We can see that

$$y' = \frac{x+y}{x-y} = \frac{1+(y/x)}{1-(y/x)}.$$

We substitute $u = y/x$, which means also that $y = ux$ and then $y' = u'x + u =$

$(1+u)/(1-u)$, which implies that

$$u'x = \frac{1+u}{1-u} - u = \frac{1+u^2}{1-u},$$

a separable equation. Separating the variables yields

$$\frac{1-u}{1+u^2}du = \frac{dx}{x},$$

and then integration gives $\arctan u - \ln(1+u^2)/2 = \ln|x| + c$. Substituting $u = y/x$ back into this expression and using that

$$-\ln(1+(y/x)^2)/2 - \ln|x| = -\ln(|x|\sqrt{1+(y/x)^2}) = -\ln(\sqrt{x^2+y^2})$$

we obtain that the solution is $\arctan(y/x) - \ln(\sqrt{x^2+y^2}) = c$.

30. The equation is *homogeneous*. (See Section 2.2, Problem 30) Algebraic manipulations show that it can be written in the form

$$y' = \frac{3y^2 + 2xy}{2xy + x^2} = \frac{3(y/x)^2 + 2(y/x)}{2(y/x) + 1}.$$

Substituting $u = y/x$ gives that $y = ux$ and then

$$y' = u'x + u = \frac{3u^2 + 2u}{2u + 1},$$

which implies that

$$u'x = \frac{3u^2 + 2u}{2u + 1} - u = \frac{u^2 + u}{2u + 1},$$

a separable equation. We obtain that $(2u + 1)du/(u^2 + u) = dx/x$, which in turn means that $\ln(u^2 + u) = \ln|x| + \tilde{c}$. Therefore, $u^2 + u = cx$ and then substituting $u = y/x$ gives us the solution $(y^2/x^3) + (y/x^2) = c$.

31. The equation can be made *exact* by choosing an appropriate integrating factor. We can check that $(M_y - N_x)/M = -(3x^2 + y)/(y(3x^2 + y)) = -1/y$ depends only on y, so $\mu(y) = e^{\int (1/y)dy} = e^{\ln y} = y$ is an integrating factor. After multiplication, the equation becomes $(3x^2y^2 + y^3)dx + (2x^3y + 3xy^2)dy = 0$. This equation is exact now, because $M_y = 6x^2y + 3y^2 = N_x$. Integrating M with respect to x gives that $\psi(x, y) = x^3y^2 + y^3x + g(y)$, then $\psi_y = 2x^3y + 3y^2x + g'(y) = 2x^3y + 3xy^2$, which means that $g'(y) = 0$, so we obtain that $g(y) = 0$ is acceptable. Therefore the general solution is defined implicitly as $x^3y^2 + xy^3 = c$. The initial condition gives us $4 - 8 = c = -4$, and the solution is $x^3y^2 + xy^3 = -4$.

32. This is a *Bernoulli* equation. (See Section 2.4, Problem 27) If we substitute $u = y^{-1}$, then $u' = -y^{-2}y'$, so $y' = -u'y^2 = -u'/u^2$ and the equation becomes $-xu'/u^2 + (1/u) - e^{2x}/u^2 = 0$, and then $u' - u/x = -e^{2x}/x$, which is a linear equation. The integrating factor is $e^{-\int (1/x)dx} = e^{-\ln x} = 1/x$, and we obtain that $(u'/x) - (u/x^2) = (u/x)' = -e^{2x}/x^2$. The integral of the function on the right hand side can not be expressed in a closed form using elementary functions, so we have to express the solution using integrals. Let us integrate both sides of this equation from 1 to x. We obtain that the left hand side turns into

$$\int_1^x (u(s)/s)'ds = (u(x)/x) - (u(1)/1) = \frac{1}{yx} - \frac{1}{y(1)} = \frac{1}{yx} - 1/2.$$

The right hand side gives us $-\int_1^x [e^{2s}/s^2]\, ds$. So we find that

$$1/y = -x\int_1^x [e^{2s}/s^2]\, ds + (x/2).$$