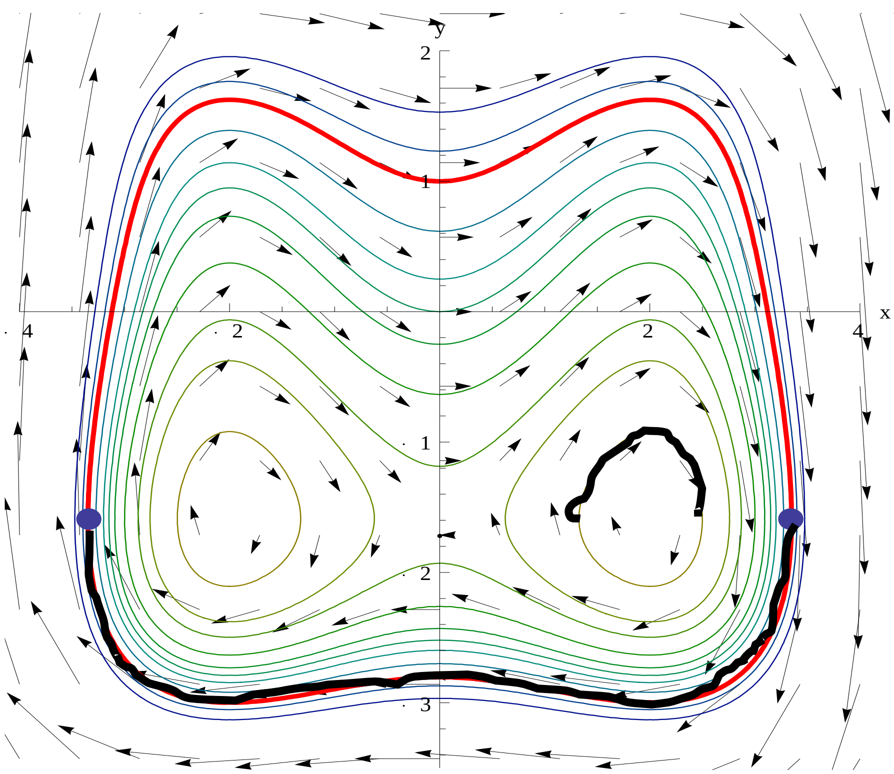
Exam is 1 hour 30 minutes. Starts at 2:15, ends at 3:45
Ok: handwritten notes, calculator (TI 83/84) **Not ok**: printouts, book, **TI 89,** laptop, tablet, cell phone or other handheld

Part I Chapters 1 and 2 (30%, 25 min)

A. There are 2 theorems which apply to first order ODE's. For the linear equation $y' + p(t)y = q(t)$ as long as the function(s)__p and q____ is (are) continuous, a solution always exists and is unique. In fact, we can often determine a minimal interval of validity. For instance, for $y' - \frac{3}{t^2-9}y = \frac{1}{t^2}$; $y(-1) = 2$, the solution has ____**(−3, 0)**___ as its interval of validity. For a nonlinear IVP $y' = f(t, y)$, $y(t_0) = y_0$, a solution always exists and is unique provided that **f** and the partial derivative of **f with respect to y** are **continuous** at the point **(t_0, y_0)**.



For example, a solution will always exist and be unique for the nonlinear equation

$$y' = \frac{4x - x^3}{4 + y^3}$$ provided that $y \neq$ **−4^(1/3)**.

The interval of validity is much more complicated as it will vary for each initial value. The slope field for the same equation is given to the left. For the IV (0,−2.8), **darken the solution on the graph**. The approximate interval of validity for the solution is __ **(−3.4, 3.4)**_. For the IV (2,−1), **darken its solution on the graph**. The approximate interval of validity of this 2nd solution is __**(1.3, 2.4)**_.

B. For fluid stability equation: $y' = (\cos t + 2)y - y^3$, identify type and provide 1st few steps of sol'n as follows:

- **Bernoulli:** perform the substitution and put resulting linear equation into standard form.

This is a **Bernoulli** equation with $n = 3$. To transform into a linear equation, make the transformation $v = y^{1-n} = y^{1-3} = y^{-2}$
Solving for y and taking the derivative, we get: $y = v^{-1/2}$ and $y' = -\frac{1}{2}v^{-3/2}v'$

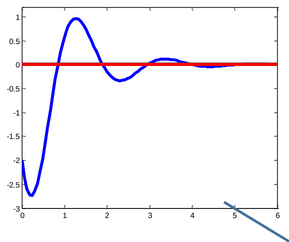
Substituting, we get $-\frac{1}{2}v^{-3/2}v' = (\cos t + 2)v^{-1/2} - v^{-3/2}$ or $v' = -2(\cos t + 2)v + 2$ or $v' + 2(\cos t + 2)v = 2$

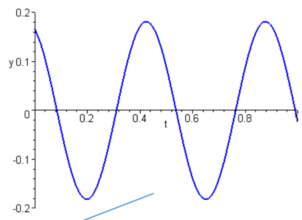
which is standard form for a linear equation.

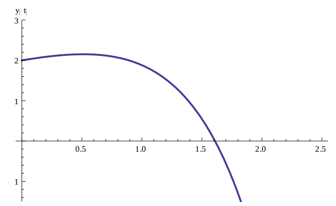
Part II Chapters 3 and 6 (40%, 35 min)

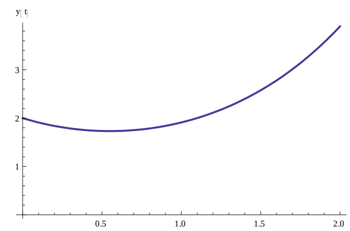
Match the following by drawing a line from each item in a row to an item in the row below.
Categories are i. graph, ii. sol'n, iii. short term analysis iv. long term analysis.

i.









ii. $y = \frac{1}{6}\cos 8\sqrt{3}t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$ $y(t) = e^{-t}(-2\cos(3t) - 3\sin(3t))$ $y(t) = 1/2\, e^{t} + 3/2\, e^{-t}$ $y(t) = -1/2\, e^{3t/2} + 5/2\, e^{t/2}$

iii. Solution is oscillatory but damped. | At first the solution has exponential decay. | Solution is purely oscillatory. | At first the solution increases.

iv. Function goes to 0 as t goes to ∞. | Function has no limit as t goes to ∞. | The exponential growth term eventually takes over and the function goes to ∞ as t goes to ∞. | Term with negative coefficient takes over and function goes to -∞ as t goes to ∞.

Given the RLC circuit with resistance 4 Ohms, inductance 1 Henry and capacitance 0.025 Farads

A. but with no external electromotive force,
   1. find the form of the solution

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0 \quad \Rightarrow \quad Q''(t) + 4Q'(t) + \frac{1}{.025}Q(t) = 0$$

$$\Rightarrow \quad Q''(t) + 4Q'(t) + 40Q(t) = 0 \text{ or } y''(t) + 4y'(t) + 40y(t) = 0$$

$$b^2 - 4ac = 4^2 - 4\cdot 40 = -144 \quad \Rightarrow \quad \omega = \sqrt{144}/2 = 6$$

   2. find natural and quasi periods and use example to discuss the effect of damping on period
      natural = 2π/√40 = .994 and quasi = 2π/6 = 1.047, about a 5% increase. In general, damping increases the period.

B. for the external forcing function E(t) = cos(6t), using the method of undetermined coefficients, find the form of a particular solution. The form of a particular solution is Acos(6t) + Bsin(6t).

C. for y(0) = 40 and y'(0) = 0 and the External forcing function $40u_3(t)$, use the Laplace transform methods to solve

$$y''(t) + 4y'(t) + 40y(t) = 40u_3(t)$$

$$s^2Y(s) - 40s + 4(sY(s) - 40) + 40Y(s) = 40e^{-3s}/s$$

$$(s^2 + 4s + 40)Y(s) = 40s + 160 + 40e^{-3s}/s$$

$$Y(s) = \frac{40s + 160}{s^2 + 4s + 40} + \frac{40e^{-3s}}{s(s^2 + 4s + 40)}$$

Next step is to use expand the 2 fractions, ignoring for the moment the exponential:

$$\frac{40s+160}{s^2+4s+40} = A\frac{s+2}{(s+2)^2+6^2} + B\frac{6}{(s+2)^2+6^2}$$

$40s+160 = A(s+2)+6B$ setting $s=-2 \Rightarrow 80=6B$ or $B=40/3$

looking at the coefficients of s on each side gives $A=40$

$$\frac{40}{s(s^2+4s+40)} = C\frac{s+2}{(s+2)^2+6^2} + D\frac{6}{(s+2)^2+6^2} + E\frac{1}{s}$$

$40 = Cs(s+2)+6Ds+E((s+2)^2+6^2)$ setting $s=0 \Rightarrow E=1$

looking at the coefficients of s^2 on each side gives $0=C+E \Rightarrow C=-1$

setting $s=-2$ gives $40=-12D+36E \Rightarrow D=3$

Applying the inverse Laplace transform:

$$Y(s) = 40\frac{s+2}{(s+2)^2+6^2} + \frac{40}{3}\frac{6}{(s+2)^2+6^2} + \left(-\frac{s+2}{(s+2)^2+6^2} + 3\frac{6}{(s+2)^2+6^2} + \frac{1}{s}\right)e^{-3s}$$

$$y(t) = e^{-2t}(40\cos 6t + (40/3)\sin 6t) + u_3(t)(-e^{-2t}\cos 6(t-3) + e^{-2t}3\sin 6(t-3)+1)$$

Part III Chapter 5 (15%, 15 min) find the recursive formula and the first few terms of a series solution to

$$(t+1)y''-ty'-y=0, \quad t_0=0$$

a. Find the recurrence relation of a power series solution. Substituting the form of the solution and its derivatives into the equation we get:

$$y(t)=\sum_{n=0}^{\infty} a_n t^n, \quad y'(t)=\sum_{n=1}^{\infty} n a_n t^{n-1}, \quad y''(t)=\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$$

$$(t+1)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=1}^{\infty} n a_n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n = 0$$

Rewrite first series as 2 series.

$$t\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=1}^{\infty} n a_n t^{n-1} - \sum_{n=0}^{\infty} a_n t^n = 0$$

Incorporate the coefficients of t into the series and shift the indices appropriately (1st by one and the 2nd by two):

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=1}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{n=1}^{\infty} (n+1)n a_{n+1} t^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=1}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n = 0$$

Note that does no harm to have the 1st and 3rd series start at 0 since the additional terms will be 0.

$$\sum_{n=0}^{\infty} (n+1)n a_{n+1} t^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - \sum_{n=0}^{\infty} n a_n t^n - \sum_{n=0}^{\infty} a_n t^n = 0$$

Extract the coefficients of t^n in each series to get recurrence relation:

$$(n+1)n a_{n+1} + (n+2)(n+1)a_{n+2} - n a_n - a_n = 0 \quad \text{or} \quad a_{n+2} = \frac{(n+1)(a_n - n a_{n+1})}{(n+2)(n+1)} = \frac{a_n - n a_{n+1}}{n+2}$$

b. Use the relation to find the first 3 terms of one basic solution.
We find 3 terms of basic solution with even exponents (set a_1=0). We apply recursion twice:

$$a_2 = \frac{a_0 - 0\cdot 0}{(0+2)} = \frac{a_0}{2}; \quad a_3 = \frac{0 - 1\cdot a_2}{(1+2)} = -\frac{a_2}{3} = -\frac{a_0}{6} \quad \Rightarrow \quad y_1(t) = a_0\left(1 + \frac{1}{2}t - \frac{1}{6}t^2 + ...\right)$$

Part IV Chapter 8 (15%, 15 min) Write down the pseudo-code for the improved Euler method with h=.05 and final value t = 2. Perform the first 2 iterations by hand and compare those values with the exact solution.

$$y' = .5 - t + 2y, \quad y(0) = 1$$

Solve the homogeneous equation to get:

$y = ce^{2t}$

Now use the method of undetermined coefficients:

$y_p = At + B \Rightarrow A = .5 - t + 2At + B \Rightarrow A = .5, B = 0$

Hence $y = ce^{2t} + .5t$ and the initial conditions $\Rightarrow c = 1$

Thus $y = e^{2t} + .5t$

- Step 1. Define $f(t, y)$=.5-t+2y
- Step 2. Input initial values t=0 and y=1
- Step 3. Input step size h=.05 and number of steps n=(tl-t0)/.05=(2-0)/.05=40
- Step 4. Output t and y
- Step 5. For j from 1 to n do

  $k1 = f(t, y)$
  $t = t + h$
  $k2 = f(t, y + h*k1)$
  $y = y + (h/2)*(k1 + k2)$
  Output t and y

- Step 7. End

First iteration:

k1=0.5-0+2*1=2.5
t=0+0.05=0.05
k2= f (0.05, 1 + .05*2.5)=f(0.05,1.125)= 0.5-0.05+2*1.125=2.7
y =1 + (.05/2)*(2.5+2.7)=1.13
Exact is $e^{2(.05)} + .5(.05) = 1.13017$

Second iteration:

k1=0.5-0.05+2*1.13=2.71
t=0.05+0.05=0.1
k2= f (0.1, 1.13 + .05*2.71)=f(0.1, 1.2655)= 0.5-0.1+2*1.2655=2.931
y =1.13 + (.05/2)*(2.71+2.931)= 1.2710
Exact is $e^{2(.1)} + .5(.1) = 1.2714$

t	exact y	euler	improved euler
0	1.0000	1.0000	1.0000
0.0500	1.1302	1.1250	1.1300
0.1000	1.2714	1.2600	1.2710

As you can see from the chart produced using MATLAB, the improved Euler really is an improvement.