Exam is 1 hour 15 minutes. All questions involve linear 2nd order equations.
Ok: handwritten notes, calculator (TI 83/84)
Not ok: printouts, book, **TI 89,** laptop, tablet, cell phone or other handheld
Each problem (or part) is worth 20% and should take about 15 minutes to complete.

1. Fill in the blank, multiple choice, true or false:
   a. Given the differential equation

$$y'' + \frac{4}{t^2 - 3t - 4} y' - \frac{3}{t^2} y = \frac{1}{t^2 - 9}; \quad y(-2) = 3,\ y'(-2) = -3$$

   Thm 3.2.1 guarantees that a solution exists and is unique and is valid for the interval (-3,-1) .

   b. Given 2 solutions to a homogeneous equation, the sum, product, difference, quotient (circle all that apply) of the two solutions is also a solution.

   c. Reduction of order was used to find a 2nd solution to a homogeneous equation with constant coeffs whose characteristic equation had distinct real, complex, repeated (circle one) roots.

   d. T or F (explain why). Given a critically damped mass-spring system, the nature of the solutions will drastically change if the damping is increased.

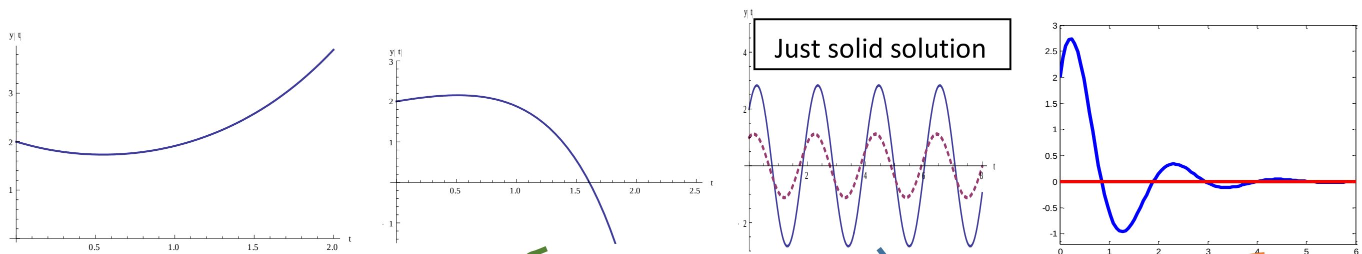
F: If damping increases, system becomes overdamped & such solns are not too different from crtically damped

   e. Match the following by drawing a line from each item in the top row to an item below:
   mass spring stiffness damping

   capacitance resistance inductance

   f. Match the following by drawing a line from each item in a row to an item in the row below. Categories are i. graph, ii. sol'n, iii. short term analysis iv. long term analysis.

i.



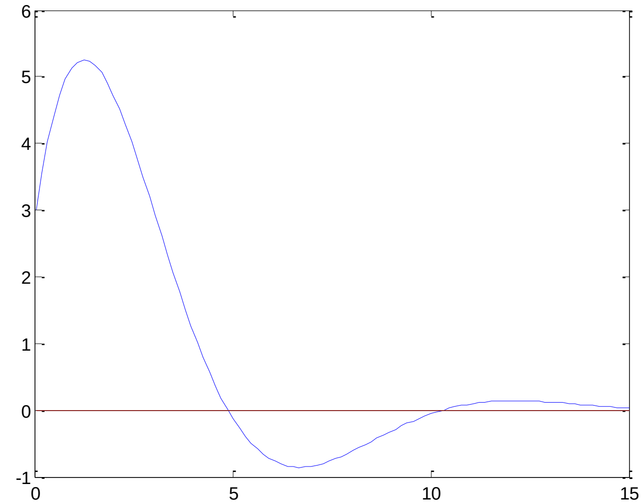
ii. $y(t) = -1/2\, e^{3t/2} + 5/2\, e^{t/2}$ $y(t) = 1/2\, e^{t} + 3/2\, e^{-t}$ $y(t) = e^{-t}(2\cos(3t) + 3\sin(3t))$ $y = 2\cos(3t) + 2\sin(3t)$

iii. Solution is oscillatory but damped. At first the solution has exponential decay. At first the solution increases. Solution is purely oscillatory.

iv. The exponential growth term eventually takes over and the function goes to ∞ as t goes to ∞. Function has no limit as t goes to ∞. Function goes to 0 as t goes to ∞. Term with negative coefficient takes over and function goes to -∞ as t goes to ∞.

2. Solve the homogeneous IVP with constant coefficients:

$$9y'' + 6y' + 4y = 0; \quad y(0) = 3,\ y'(0) = 4$$



$$r = \frac{-6 \pm \sqrt{6^2 - 4\cdot 9\cdot 4}}{2\cdot 9} = \frac{-6 \pm \sqrt{36(1-4)}}{2\cdot 9}$$

$$= \frac{-6 \pm \sqrt{36(1-4)}}{2\cdot 9} = -\frac{1}{3} \pm \frac{\sqrt{3}}{3} i$$

$$y = e^{-(1/3)t}\left(c_1 \cos\left(t/\sqrt{3}\right) + c_2 \sin\left(t/\sqrt{3}\right)\right)$$

$$y' = e^{-(1/3)t}\left(c_1\left(-(1/3)\cos\left(t/\sqrt{3}\right) - \left(1/\sqrt{3}\right)\sin\left(t/\sqrt{3}\right)\right)\right.$$

$$\left. + c_2\left(-(1/3)\sin\left(t/\sqrt{3}\right) + \left(1/\sqrt{3}\right)\cos\left(t/\sqrt{3}\right)\right)\right)$$

$c_1 = 3$ and $-c_1/3 + c_2/\sqrt{3} = 4 \quad \Rightarrow \quad c_2 = 5\sqrt{3}$

$$y = e^{-(1/3)t}\left(3\cos\left(t/\sqrt{3}\right) + 5\sqrt{3}\sin\left(t/\sqrt{3}\right)\right)$$

3. Given that one solution is $y_1(x) = x^3$ use reduction of order to find a 2[nd] solution to the homogeneous eqtn:

$$x^2 y'' - 5xy' + 9y = 0; \quad x > 0$$

Assume 2[nd] soln is multiple of 1[st]:

$$y_2(x) = v(x)x^3$$

$$y_2'(x) = v'(x)x^3 + 3v(t)x^2$$

$$y_2''(x) = v''(x)x^3 + 6v'(x)x^2 + 6v(x)x$$

Substituting these into the ODE and collecting terms:

$$x^2(v''(x)x^3 + 6v'(x)x^2 + 6v(x)x) - 5x(v'(x)x^3 + 3v(t)x^2) + 9v(x)x^3 = 0$$

$$v''(x)x^5 + 6v'(x)x^4 + 6v(x)x^3 - 5v'(x)x^4 - 15v(t)x^3 + 9v(x)x^3 = 0$$

$$v''(x)x^5 + v'(x)x^4 = 0 \text{ or } v''(x)x + v'(x) = 0$$

Let $u = v'$, then $xu' = -u$ and $\frac{du}{u} = -\frac{dx}{x}$, so $u = c/x$ and $v = c\ln x + d$

So a 2[nd] solution is $y_2(x) = x^3 \ln x$

4. Find the form of a general solution **but do not solve** the non-homogeneous eqtn with constant coefficients:

$$y'' + 3y' = 3e^{-3t} + 3 + 2t - 2e^{2t}\cos 3t$$

$r^2 + 3r = 0$ so $r = 0, -3$ and $y_h = c_1 + c_2 e^{-3t}$

Divide the RHS into 3 parts: $\underset{p_1}{3e^{-3t}} + \underset{p_2}{3 + 2t} - \underbrace{2e^{2t}\cos 3t}_{p_3}$

$$y_{p_1} = tAe^{-3t}, \quad y_{p_2} = t(B + Ct), \quad y_{p_3} = De^{2t}\cos 3t + Ee^{2t}\sin 3t$$

Note how the first 2 particular solution forms have been multiplied by *t* so that the homogeneous solution doesn't "interfere". Putting all soln forms together, we get that general solution will have following form:

$$y = y_h + y_{p_1} + y_{p_2} + y_{p_3} = c_1 + c_2 e^{-3t} + tAe^{-3t} + t(B + Ct) + De^{2t}\cos 3t + Ee^{2t}\sin 3t$$

5. Given the RLC circuit with resistance 16 Ohms, inductance 2 Henries and capacitance 0.02 Farads but with no external electromotive force, determine the associated ODE for the charge on the capacitor and classify the nature of the damping. If underdamped, compare the quasi-period with the period for the same circuit but with the resistor removed. In any case determine the value of the resistor (with the same inductor and capacitor) that produces a circuit that is critically damped.

$$LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = 0 \quad \Rightarrow \quad 2Q''(t) + 16Q'(t) + \frac{1}{.02}Q(t) = 0$$

$$\Rightarrow \quad 2Q''(t) + 16Q'(t) + 50Q(t) = 0 \quad \Rightarrow \quad Q''(t) + 8Q'(t) + 25Q(t) = 0$$

$$b^2 - 4ac = 8^2 - 4\cdot 25 = -36 \quad \Rightarrow \quad \omega = \sqrt{36}/2 = 3$$

So the circuit is underdamped and the quasi period is $2\pi/3$

In contrast, the natural period is $2\pi/5$. The critical resistor for damping satisfies

$$b^2 - 4ac = R^2 - 4L/C = 0 \text{ or } R^2 - 4\cdot 2/0.02 = 0 \text{ or } R^2 = 400 \text{ or } R = 20\Omega$$