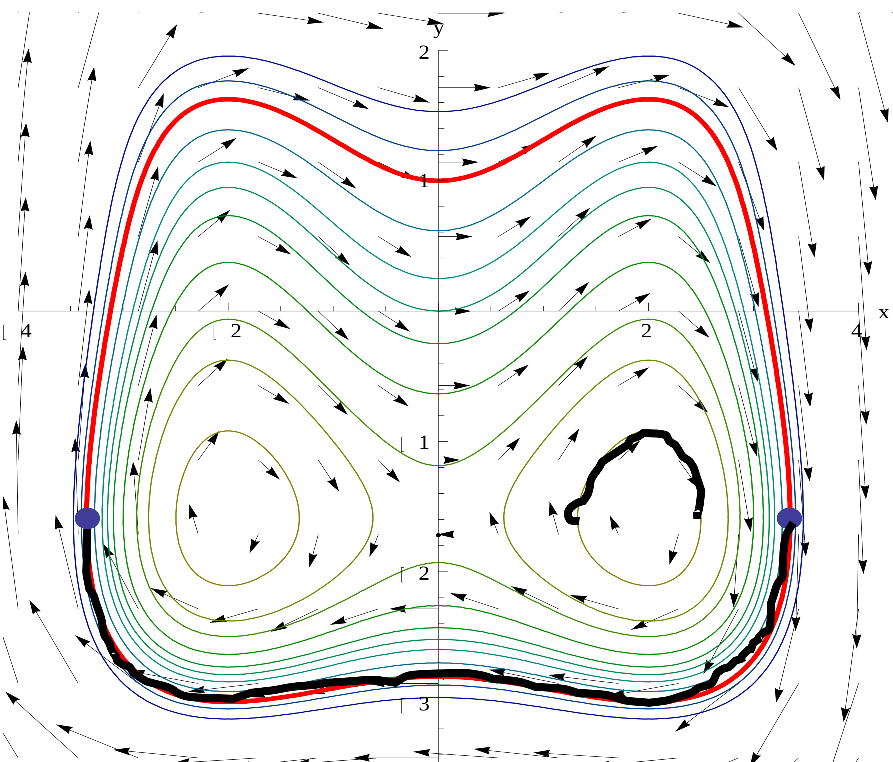
Exam is 1 hour 15 minutes.
Ok: handwritten notes, calculator (TI 83/84) **Not ok**: printouts, book, **TI 89,** laptop, tablet, cell phone or other handheld

Part I (30%, 10 min)

There are 2 theorems which apply to first order ODE's. For a nonlinear IVP $y' = f(t, y)$, $y(t_0) = y_0$, a solution always exists and is unique provided that __**f**____ and the partial derivative of ___**f**____ are ___**continuous**__ at the point ______(t_0, y_0)______. For the linear equation $y' + p(t)y = q(t)$ as long as the function(s)__**p & q**__ is (are) continuous, a solution always exists and is unique. In fact, we can often determine a minimal interval of validity. For instance, for $y' - \frac{3}{t^2 - 9}y = \frac{1}{t^2}$; $y(-1) = 2$, the solution has ____**(−3, 0)**____ as its interval of validity. For the nonlinear equation



$y' = \frac{4x - x^3}{4 + y^3}$ a solution will always exist and be unique provided that $y \neq$ **−4^(1/3)_.** The interval of validity is much more complicated as it will vary for each initial value. The slope field for the same equation is given to the left. For the IV (0,−2.8), **darken the solution on the graph**. The approximate interval of validity for the solution is _____**(−3.4, 3.4)**____. For the IV (2,−1), **darken its solution on the graph**. The approximate interval of validity of this 2nd solution is ____**(1.3, 2.4)**____. Assuming that the solutions follow the direction of the arrows, for any solution whose initial value is in **Q1 (quadrant I),** the limiting behavior of the solution is $\lim_{x \to a} y = -\sqrt[3]{4}$, where _$a > 2.7$_ is variable (depends on the initial value).

Work for Part I:

Part II (15%, 15 min) Solve the IVP and provide a minimal interval of validity (domain) for the solution:

$$t^2 y' + 3y = -2t^3, \quad y(-1) = 0,$$

$$y' + \frac{3}{t^2} y = -2t \Rightarrow \mu = e^{\int 3/t^2 dt} = e^{-3/t}$$

$$ye^{-3/t} = -2\int te^{-3/t} dt \text{ or } y = -2e^{3/t} \int te^{-3/t} dt$$

We now incorporate the initial value into the solution:

$$y = -2e^{3/t} \int_{-1}^{t} ue^{-3/u} du$$

Note that the integral will always be 0 at the initial t-value. Hence, if the initial y-value had been nonzero, then there would have been a constant as well. For example, if

$$t^2 y' + 3y = -2t^3, \quad y(-1) = 3,$$

then the solution would have been

$$y = e^{3/t} \left(-2\int_{-1}^{t} ue^{-3/u} du + 3e^3 \right)$$

Part III (15%, 15 min)

- **a)** Verify that $xy + y^2 + (x^2 + 3xy)y' = 0$ is not exact.
- **b)** Use y as an IF (integrating factor) and show that the new equation is exact.
- **c)** Solve the equation to get an implicit solution.

a)

$$M = xy + y^2 \text{ and } N = x^2 + 3xy$$

$$M_y = x + 2y \neq 2x + 3y = N_x \text{ so not exact}$$

b) Multiplying by IF = y:

$$xy^2 + y^3 + (x^2 y + 3xy^2) y' = 0$$

and

$$M = xy^2 + y^3 \text{ and } N = x^2 y + 3xy^2$$

$$M_y = 2xy + 3y^2 = N_x \text{ so exact!}$$

c) Integrating M with respect to x:

$$\psi = \int Mdx = \int xy^2 + y^3 dx = \frac{x^2 y^2}{2} + xy^3 + C(y)$$

$$\psi_y = x^2 y + 3xy^2 + C'(y) = N = x^2 y + 3xy^2$$

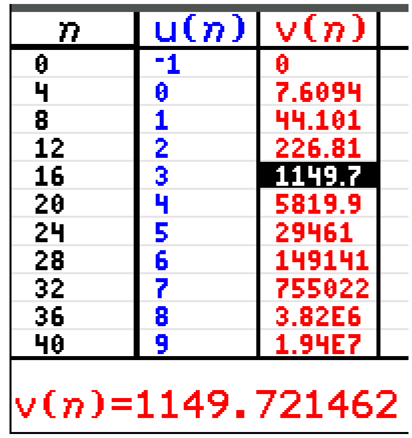
Thus, we can choose $C(y) = 0$ and an implicit solution is

$$\frac{x^2 y^2}{2} + xy^3 = c' \text{ or } x^2 y^2 + 2xy^3 = c$$

Part IV (10%, 10 min) Euler method for numerical solution: $h = 0.25$; t final is 3. Be sure to provide a screen shot for the "y=" window in "seq" mode as well a table showing the values of t and y for integer values of t.

$y' = 3 - t + 2y; \quad y(-1) = 0$

nMin =	**0**
u(n) =	**u(n−1)+.25**
u(nMin) =	**−1**
v(n) =	**v(n−1)+.25(3−u(n−1)+2 v(n−1))**
v(nMin) =	**0**



n	u(n)	v(n)
0	-1	0
4	0	7.6094
8	1	44.101
12	2	226.81
16	3	1149.7
20	4	5819.9
24	5	29461
28	6	149141
32	7	755022
36	8	3.82E6
40	9	1.94E7

v(n)=1149.721462

Notice that the y final has been found to 6 decimal digits, even though only one decimal digit is displayed in the table.

Also note that TblStart and ΔTbl have been set to 0 and 4, respectively. (4 is the reciprocal of .25). These ensure that only the points with integer values of u (which is t) are displayed.

Part V (30%, 20 min) For each equation, identify its type and provide the 1st few steps of its sol'n as follows:

- **Separable:** separate variables and express as integrals. Note what integration technique to use to evaluate the integrals (integration by parts, partial fractions, etc.).
- **Homogeneous:** perform the substitution and separate the variables.
- **Linear:** put in standard form and find and apply the IF μ. Express y as an integral. Note what integration technique to use to evaluate the integrals (integration by parts, partial fractions, etc.).
- **Bernoulli:** perform the substitution and put resulting linear equation into standard form.
- **Exact:** show exactness, integrate M with respect to x to find Ψ up to "constant" C(y)

1. $xy + y^2 + (x^2 + 3xy)y' = 0$

This is the same equation that was tackled in part III. Here we treat it as **homogeneous**.

$$y' = -\frac{xy + y^2}{x^2 + 3xy} = -\frac{(y/x) + (y/x)^2}{1 + 3(y/x)} = -\frac{v + v^2}{1 + 3v}$$

Since $v=y/x$, $y=vx$ and

$$y' = v'x + v = -\frac{v + v^2}{1 + 3v} \text{ or } v'x = -\frac{v + v^2}{1 + 3v} - v = -\frac{v + v^2}{1 + 3v} - v\frac{1 + 3v}{1 + 3v} = -\frac{2v + 4v^2}{1 + 3v}$$

so

$$\frac{dv}{dx}x = -2\frac{2v^2 + v}{3v + 1} \text{ or } \frac{3v + 1}{2v^2 + v}dv = \frac{-2}{x}dx$$

2. $e^{2y} - (t^2 - 9)\sin(3y)\,y' = 0$

This is a **separable** equation:

$$e^{2y} = (t^2 - 9)\sin(3y)\frac{dy}{dt} \text{ or } \frac{dt}{t^2 - 9} = e^{-2y}\sin(3y)\,dy$$

The LHS can be integrated using partial fractions or the hyperbolic tangent. For the RHS, use integration by parts twice.

3. Fluid stability equation: $y' = (\cos t + 2)y - y^3$

This is a **Bernoulli** equation with $n = 3$. To transform into a linear equation, make the transformation $v = y^{1-n} = y^{1-3} = y^{-2}$

Solving for y and taking the derivative, we get: $\quad y = v^{-1/2} \text{ and } y' = -\frac{1}{2}v^{-3/2}v'$

Substituting, we get $\quad -\frac{1}{2}v^{-3/2}v' = (\cos t + 2)v^{-1/2} - v^{-3/2} \text{ or } v' = -2(\cos t + 2)v + 2 \text{ or } v' + 2(\cos t + 2)v = 2$

which is standard form for a linear equation.