- You must use a graphing calculator.
- Actual exam will be about 1/3 of this practice exam in length.
- At the end of class, be sure to turn in your formula sheet (1 sheet, 2 pages, hand-written), worth 10%.

  Note that the table at the end has been corrected and will be included on the actual exam as well.

1. (10 pts) The formula **=$B$2*C4** is located in cell D3.
   a) What does cell D3 evaluate to? **3*4 = 12**
   b) If this was copied and pasted into cell **B1**, what would the resulting formula be? **=$B$2*A2 (decrement both the row and column indices that are relative by 2).**

	A	B	C	D
1	2	?????????	4	5
2	3	3	8	6
3	5	4	3	**=$B$2*C4**
4	4	3	4	9

2. (10 pts) Based on recent experience, 10-year-old passenger cars going through a motor vehicle inspection station have an 80% chance of passing the emissions test. Suppose that 200 such cars will be checked out next week.
   a. Find an expression for the expected number of cars that pass by using the definition of expectation.

$$E[X] = \sum_{k=0}^{n} k\,p(k) = \sum_{k=0}^{n} k\binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{200} k\binom{200}{k} 0.8^k 0.2^{200-k}$$

   b. Find the number of cars that are expected to pass by finding the expectation for i^{th} car X_i and using $E(\sum X_i) = \sum E(X_i)$

X_i:

x_i	0	1
$P(x_i)$	0.2	0.8

so $E(X_i) = 0.8$

and $E(X) = E(\sum_{i=0}^{200} X_i) = \sum_{i=0}^{200} E(X_i) = \sum_{i=0}^{200} 0.8 = 200 * 0.8 = 160$

3. (15 pts) A typical day's production of a certain electronic component is 12. The probability that one of these components needs rework is 0.11. Each component needing rework costs $100.
   a. What is the average daily cost for defective components?

X_i:

x_i	0	\$100
$P(x_i)$	0.89	0.11

so $E(X_i) = \$11$

Let Y be the number of components created, then E(Y) = 12 and

$$E(X) = E\left(\sum_{i=0}^{12} X_i\right) = \sum_{i=0}^{12} E(X_i) = \sum_{i=0}^{12} 11 = 12 * \$11 = \$132$$

   b. For the days when the production is exactly 12, find the standard deviation for the daily cost for the defective components.

X_i^2:

x_i^2	0	\$10000
$P(x_i)$	0.89	0.11

so $V(X_i) = E(X_i^2) - E(X_i)^2 = 1100 - 11^2 = 979$ and

$V(X) = V(\sum_{i=0}^{12} X_i) = \sum_{i=0}^{12} V(X_i) = \sum_{i=0}^{12} 979 = 12 * 979 = 11748$ and SD(X)=$\sqrt{11748} \approx \$108$

Note: the 2nd equality requires independence which we do have.

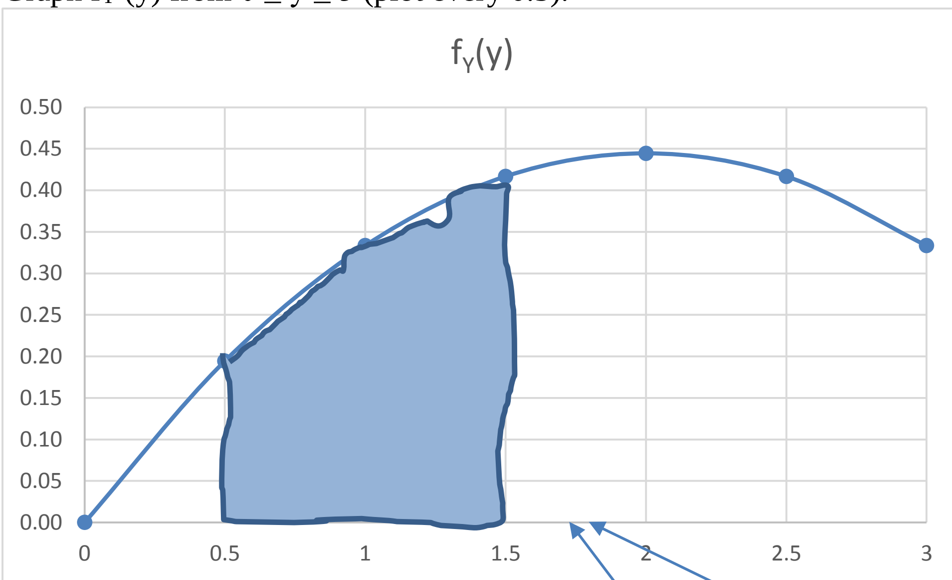
Alt. soln: N=the number of defectives is binomial, hence V(N)=npq=12(.11)(.89) ≈1.175 and SD(N)≈√ 1.175 ≈ 1.084 SD(X)=SD($100N)=|$100| SD(N)≈$100*1.08=$108

4. (35 pts) Let $f_Y(y) = ay(4-y), \quad 0 \le y \le 3$. NOTE: PDF has been changed!

a. Find a so that $f_Y(y)$ is a PDF.

$$\int_0^3 f_Y(y)dy = \int_0^3 ay(4-y)dy = a\left(2y^2 - \frac{y^3}{3}\right)\Bigg|_{y=0}^{3} = a\left(18 - \frac{27}{3}\right) - (0-0) = 9a \text{ so a=1/9}$$

b. Graph $f_Y(y)$ from $0 \le y \le 3$ (plot every 0.5).

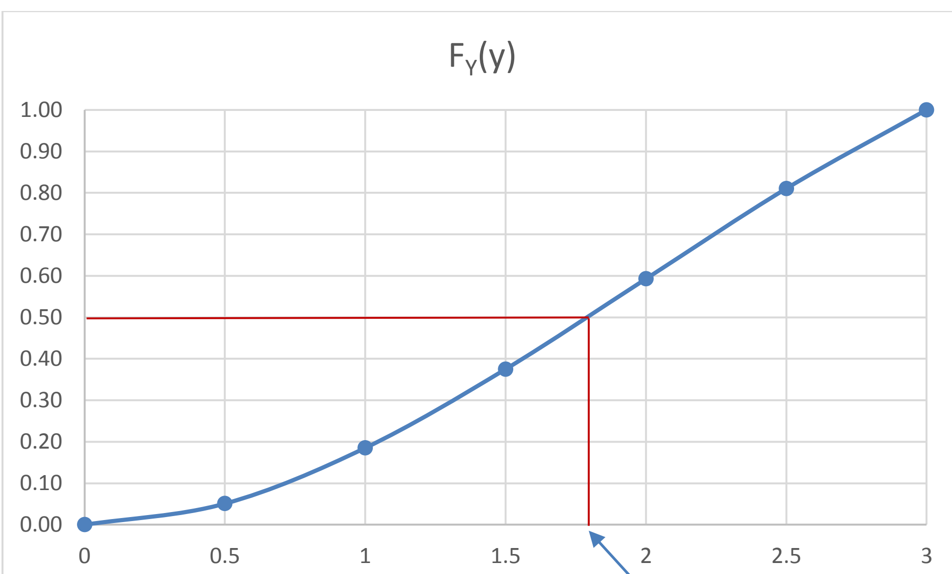


y	0	0.5	1	1.5	2	2.5	3
$f_Y(y)$	0.00	0.19	0.33	0.42	0.44	0.42	0.33

mean=1.75 median=1.79

c. Find and graph $F_Y(y)$, the corresponding CDF, from $0 \le y \le 3$ (plot every 0.5).

$$F_Y(y) = \int_0^y f_Y(t)dt = \frac{1}{9}\int_0^y t(4-t)dt = \frac{1}{9}\left(2t^2 - \frac{t^3}{3}\right)\Bigg|_{t=0}^{y} = \frac{1}{9}\left(2y^2 - \frac{y^3}{3}\right) = \frac{y^2}{27}(6-y)$$



y	0	0.5	1	1.5	2	2.5	3
$F_Y(y)$	0.00	0.05	0.19	0.38	0.59	0.81	1.00

median=1.79

d. Find $P(|Y-1|<0.5)$. Graph the inequality on your plot from part b) and shade the area representing the probability.

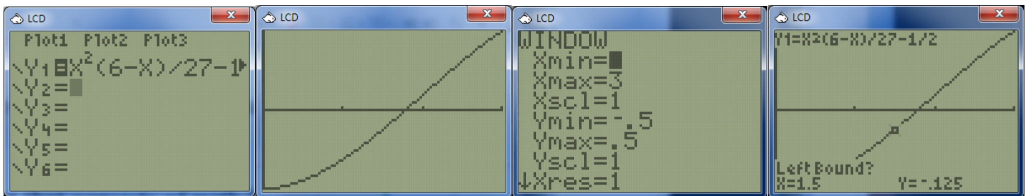
$$P(|Y-1|<0.5)=F_Y(1.5)-F_Y(0.5)=\frac{(1.5)^2}{27}(6-1.5)-\frac{(0.5)^2}{27}(6-0.5)\approx 0.32$$

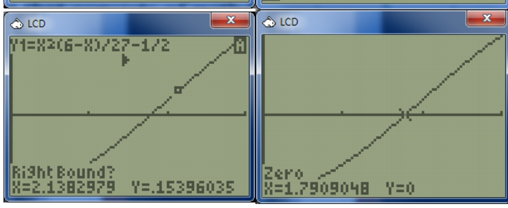
e. Find the expectation and label on your plot from part b)

$$E(Y)=\int_0^3 y\,f_Y(y)dy=\int_0^3 y^2(4-y)dy=\frac{1}{9}\left(\frac{4y^3}{3}-\frac{y^4}{4}\right)\Bigg|_{y=0}^{3}=\frac{1}{9}\left(\frac{4*3^3}{3}-\frac{3^4}{4}\right)=\frac{1}{9}\left(\frac{36}{1}-\frac{81}{4}\right)=\frac{7}{4}$$

f. Find the median and label on both graphs (the first with a vertical line and the second with a horizontal line and a vertical line).
To find the median, set $F_Y(y)=1/2$ and solve for y:

$$\frac{y^2}{27}(6-y)=1/2\Rightarrow \text{(use numeric equation or root finder on calculator)}$$





Hence the median≈1.79

g. Compare relative positions of the median and the expectation. Explain how comparison relates to any skewing. **The mean is to the left of the median, therefore the skewing is to the left.**

5. (40 pts) Consider an experiment that consists of withdrawing a ball from the box, NOT replacing it, and withdrawing a second ball. There are 2 red and 4 green balls in the box.

a. What is the sample space of this experiment? Is this a random variable? Why or why not?
$\{rg, gr, gg\}$ No, outcomes are not numbers.

b. To get a random variable W, we count the number of red balls selected. Find the possible outcomes and express their probabilities as quotients of binomial coefficients.

x	0	1	2
$P(x)$	$\frac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}}$	$\frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}}$	$\frac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}}$

c. Evaluate the binomial coefficients, leaving as fractions. Write W as a matrix with row 1 as the outcomes and row 2 the corresponding probabilities.

x	0	1	2
$P(x)$	$\frac{1*6}{15}=\frac{6}{15}$	$\frac{2*4}{15}=\frac{8}{15}$	$\frac{1*1}{15}=\frac{1}{15}$

d. Use the 2 row matrix of c to find the expectation μ=E(W).
Expectation is dot product of the 2 rows above thought of as vectors: 0*6/15+1*8/15+2*1/15=10/15=2/3

e. Find the 2 row matrix for W^2 and use it find $\mu_2=E(W^2)$.

x^2	0	1	4
$P(x)$	$\frac{6}{15}$	$\frac{8}{15}$	$\frac{1}{15}$

$\mu_2=E(W^2)=8/15+4/15=12/15=4/5$

f. Use the results from d and e to find V(W). $V(W) = \mu_2 - \mu^2 = 4/5 - (2/3)^2 = 36/45 - 20/45 = 16/45$

g. If X is an RV indicating whether the first ball is red or not (1, 0 respectively) and Y is whether the second ball is red or not (1, 0 respectively). Find E(X) and E(Y) and show that their sum corresponds to your answer in d.

X:

x^2	0	1
x	0	1
$P(x)$	$\frac{2}{3}$	$\frac{1}{3}$

& Y:

y^2	0	1
y	0	1
$P(y)$	$\frac{2}{3}$	$\frac{1}{3}$

so E(X)=E(Y)=1/3 and so E(X+Y)= E(X)+E(Y)=1/3+1/3=2/3

h. Find V(X) and V(Y) for X and Y described in f. and show that their sum DOES NOT correspond to your answer in d. Why not? Try to reason why it is more or less.
$V(X)=V(Y)=E(X^2)-E(X)^2=1/3-(1/3)^2=(3-1)/9=2/9$
However, note that $16/45=V(X+Y)\neq V(X)+V(Y) = 2/9 + 2/9=4/9=20/45$
X and Y are dependent so V(X+Y)=V(X)+V(Y) does not hold. Basically, the variability in the result gets reduced because of the dependence.

6. (25 pts) Two chips are drawn at random and without replacement from an urn that contains five chips, numbered 1 through 5. If the sum of the chips drawn is even, the random variable X = 5; if the sum of the chips drawn is odd, X = −3.

a. Use a 5 x 5 grid with the diagonal filled in to find the possible sums of the chips. Circle those

	1	2	3	4	5
1		3	4	5	6
2	3		5	6	7
3	4	5		7	8
4	5	6	7		9
5	6	7	8	9	

x	-3	5
$P(x)$	$\frac{12}{20}=\frac{3}{5}$	$\frac{8}{20}=\frac{2}{5}$

b. Using the definition, find the moment-generating function for X.

$$M_X(t) = E[e^{tX}] = \sum_k P(X=k) * e^{kt} = \frac{3}{5}e^{-3t} + \frac{2}{5}e^{5t} = \frac{3e^{-3t}+2e^{5t}}{5}$$

c. Write X as a linear function of a Bernoulli distribution.

If B:

b	0	1
$P(b)$	$\frac{3}{5}$	$\frac{2}{5}$

, then $X = (5 - -3)B - 3 = 8B - 3$

d. Use the table to find the MGF of this Bernoulli distribution.

$$M_B(t) = q + pe^t = \frac{3}{5} + \frac{2}{5}e^t$$

e. Use your results from b and c and a theorem to find the MGF of X.

Thm 3.12.3 a. Let W be a RV with MGF $M_W(t)$. Let V = aW +b. Then $M_V(t)=e^{bt}M_W(at)$

So W=B, V=X, a=8, b= −3: $M_X(t) = e^{-3t}M_B(8t) = e^{-3t}\left(\frac{3}{5}+\frac{2}{5}e^{8t}\right) = \frac{3}{5}e^{-3t} + \frac{2}{5}e^{5t}$

7. (25 pts) Let Y be a uniform random variable defined over the interval (0, 2).

a. Find a formula for the r^{th} moment of Y about the origin.

$$\mu_r = E(y^r) = \frac{1}{2}\int_0^2 y^r dt = \frac{1}{2}\left(\frac{y^{r+1}}{r+1}\right)\Bigg|_{y=0}^{2} = \frac{2^r}{r+1}$$

b. Use the formula of a. to find the first 4 moments.

r	1	2	3	4
μ_r	1	4/3	2	16/5

c. Use b. and binomial expansion to find $\sigma^2 = E[(Y-\mu)^2]$.

$\sigma^2 = E[(Y-\mu)^2] = \mu_2 - 2\mu\mu + \mu^2 = 4/3 - 2*1 + 1 = 1/3$

d. Use b. and binomial expansion to find $E[(Y-\mu)^4]$.

$E[(Y-\mu)^4] = \mu_4 - 4\mu_3\mu + 6\mu_2\mu^2 - 4\mu\mu^3 + \mu^4 = 16/5 - 4*2*1 + 6*4/3*1 - 4*1*1 + 1 = 19/5 = 0.2$

e. Use c. and d. to find the Kurtosis γ_2.

$\gamma_2 = E[(Y-\mu)^4]/\sigma^4 - 3 = 0.2/(1/9) - 3 = 1.8 - 3 = -1.2$

8. (20 pts) For the exponential pdf $f_Y(y) = 2e^{-2y}, \quad 0 < y \le \infty$

a. Find the moment generating function $M_Y(t)$ using the definition.

$M_Y(t) = E[e^{tY}] = 2\int_0^\infty e^{yt} e^{-2y} dy = 2\int_0^\infty e^{y(t-2)}\, dy = \frac{2}{t-2} e^{y(t-2)} \Big|_{y=0}^{\infty} = \frac{2}{2-t}$; provided that t<2

b. Use $M_Y(t)$ to find the first moment μ.

$M'_Y(t) = 2(2-t)^{-2}$ so $\mu = M'_Y(0) = 2/4 = 1/2$

c. Use $M_Y(t)$ to find the second moment μ_2 and the variance σ^2.

$M''_Y(t) = 4(2-t)^{-3}$ so $\mu_2 = M''_Y(0) = 4/8 = 1/2$ and $\sigma^2 = \mu_2 - \mu^2 = ½ - ¼ = ¼$

d. Use $M_Y(t)$ to find the third moment μ_3 and the skewing γ_1.

$M^{(3)}_Y(t) = 12(2-t)^{-4}$ so $\mu_3 = M^{(3)}_Y(0) = 12/16 = 3/4$

$E[(Y-\mu)^3] = \mu_3 - 3\mu_2\mu + 3\mu\mu^2 - \mu^3 = 3/4 - 3*1/2*1/2 + 3*1/2*1/4 - 1/8 = 1/4$

$\gamma_1 = E[(Y-\mu)^3]/\sigma^3 = 1/4/(1/2)^3 = 2$

9. (15 pts) Given the MGF $M_Y(t) = (.3 + .7e^t)^{10}$:

a. Use the table to identify the distribution.

B(10,.7) (binomial n=10 and p=0.7)

b. Use $M_Y(t)$ to find the first moment μ.

$M_Y'(t) = 7e^t(.3 + .7e^t)^9$ so $\mu = M_Y'(0) = 7$

c. Use $M_Y(t)$ to find the second moment μ_2 and the variance σ^2.

$M_Y''(t) = 7e^t(.3 + .7e^t)^9 + 7*6.3e^{2t}(.3 + .7e^t)^8$ so $\mu_2 = M_Y''(0) = 7(7.3) = 51.1$

and $\sigma^2 = \mu_2 - \mu^2 = 51.1 - 49 = 2.1$ (which of course is the same answer as from the npq formula)

10. (10 pts) Given the MGF $M_Y(t) = (1 - 0.4t)^{-5}$?

a. Use the table to identify the distribution.

This matches the MGF for the gamma function: $(1 - t\lambda^{-1})^{-r}$, $\lambda^{-1} = 0.4 = \frac{2}{5}$ or $\lambda = \frac{5}{2} = 2.5$. r = 5

b. Use $M_Y(t)$ to find the third moment μ_3.

$M^{(3)}_Y(t) = 5*6*7*0.4^3 (1-t)^{-8}$ so $\mu_3 = M^{(3)}_Y(0) = 5*6*7*2^3/5^3 = 16*3*7/5^2 = 13.44$

11. (40 pts) Use Theorem 3.12.3 and the table to determine which of the following statements is true. If the statement is not true, then use the table to identify the distribution.

a. The sum of two independent exponential random variables with the same parameter λ has an exponential distribution.

Theorem 3.12.3 b. stated for sum of 2 RV's X+Y is

Given $M_X(t)$ and $M_Y(t)$ then $M_{X+Y}(t) = M_X(t) * M_Y(t)$

$(1 - t\lambda^{-1})^{-1} * (1 - t\lambda^{-1})^{-1} = (1 - t\lambda^{-1})^{-2}$

which is NOT exponential. Instead it is gamma with r=2.

b. The sum of two independent binomial random variables with the same parameter *p* has a binomial distribution.

$$(q+pe^t)^n * (q+pe^t)^m = (q+pe^t)^{n+m}$$

So in this case the sum of 2 binomials with the same p is a 3rd binomial. If the first 2 have n and m trials respectively, then the sum RV has n+m trials.

c. The sum of two independent geometric random variables with the same parameter *p* has a geometric distribution.

$$\frac{p}{1-qe^t} * \frac{p}{1-qe^t} = \left(\frac{p}{1-qe^t}\right)^2$$

which is NOT geometric. Instead it is negative binomial with r=2.

d. The sum of two independent normal random variables has a normal distribution.

$$e^{\mu t+\frac{1}{2}\sigma^2 t^2} * e^{\gamma t+\frac{1}{2}\rho^2 t^2} = e^{(\mu+\gamma)t+\frac{1}{2}(\sigma^2+\rho^2)t^2}$$

Yes! $N(\mu,\sigma^2) + N(\gamma,\rho^2) = N(\mu+\gamma,\sigma^2+\rho^2)$

Distribution	$f_Y(y)$	$M_X(t)$
Binomial B(n, p)	$p(k) = \binom{n}{k} p^k q^{n-k}$	$(q+pe^t)^n$
Geometric (counting failures only)	$p(k) = pq^k$	$\frac{p}{1-qe^t}$
Negative Binomial (counting failures)	$p(k) = \binom{r+k-1}{r-1} q^k p^r$	$\left(\frac{p}{1-qe^t}\right)^r = \frac{p^r}{(1-qe^t)^r}$
Poisson	$p(k) = \lambda^k e^{-\lambda}/k!$	$e^{\lambda(e^t-1)}$
Uniform	$f_Y(y) = \frac{1}{(b-a)}; a \le y \le b$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Uniform discrete	$p(k) = \frac{1}{(b-a+1)}; a \le k \le b$	$\frac{e^{t(b+1)}-e^{ta}}{(e^t-1)(b-a+1)}$
Normal	$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$	$e^{\mu t+\frac{1}{2}\sigma^2 t^2}$
Exponential	$f_Y(y) = \lambda e^{-\lambda y}$	$(1-t\lambda^{-1})^{-1}$
Gamma	$f_Y(y) = \lambda^r/(r-1)!\, y^{r-1} e^{-\lambda y}$	$(1-t\lambda^{-1})^{-r}$