NYCCT MAT2572 Halleck fall 2017 Practice exam 2 solutions

1. Based on recent experience, 10-year-old passenger cars going through a motor vehicle inspection station have an 80% chance of passing the emissions test. Suppose that a municipality owns 200 such cars.

   a. Find the RV for the number of cars that do **not** pass by using the binomial distribution.
   Chance that k cars do **not** pass is $p(k) = \binom{200}{k} 0.2^k 0.8^{200-k}$

   b. Using your work from part a and the definition of expectation, find an expression for the expectation of the number of repairs needed.
   $$E[X] = \sum_{k=0}^{n} k\, p(k) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{200} k \binom{200}{k} 0.2^k 0.8^{200-k}$$

   c. Find the expectation for the i^{th} car **not** to pass $E(X_i)$ by finding the RV that represents the i^{th} car X_i (0 for pass, 1 for not pass) and using the definition of expectation.

x_i	0	1
$P(x_i)$	.8	.2

$$E(X_i) = 0 * .8 + 1 * .2 = .2$$

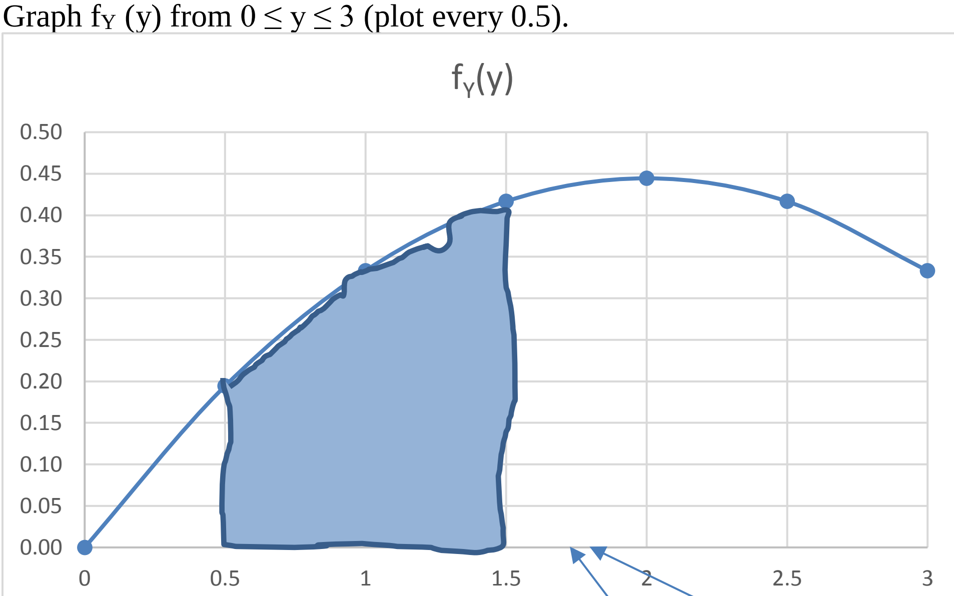
   d. Find the expectation for # cars that are **not** expected to pass by using your answer to c. and $E(\sum X_i) = \sum E(X_i) = .2 + .2 + \cdots + .2 = 200 * 0.2 = 40$

   e. If the average cost for a repair is $300, about how much can the municipality expect to spend this year on emission repairs for these 200 cars?
   $$E[\$300 * X] = \$300 E[X] = \$300 * 40 = \$12{,}000$$

2. Let $f_Y(y) = ay(4-y), \quad 0 \le y \le 3.$

   a. Find a so that $f_Y(y)$ is a PDF.
   $$\int_0^3 f_Y(y)dy = \int_0^3 ay(4-y)dy = a\left(2y^2 - \frac{y^3}{3}\right)\Bigg|_{y=0}^{3} = a\left(18 - \frac{27}{3}\right) - (0-0) = 9a \text{ so a=1/9}$$

   b. Graph $f_Y(y)$ from $0 \le y \le 3$ (plot every 0.5).

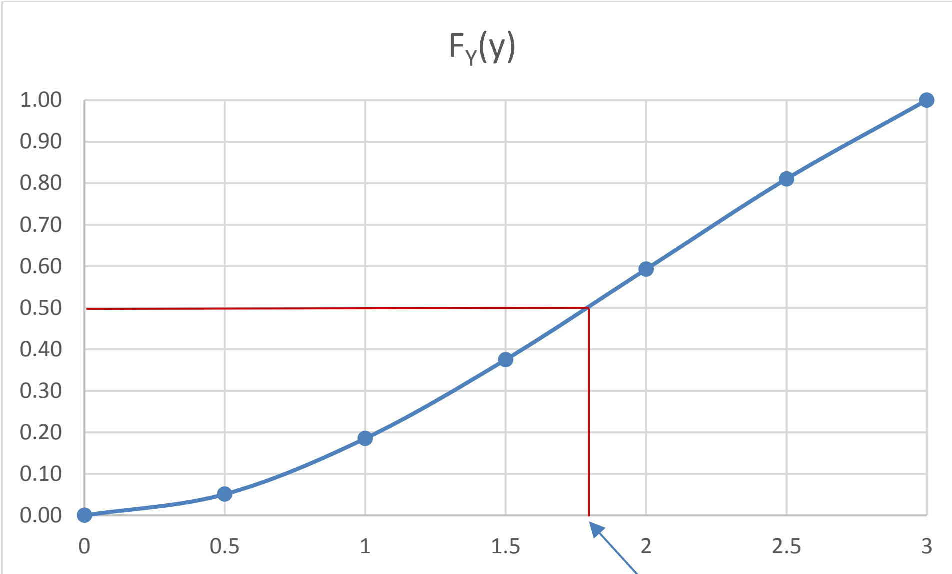


y	0	0.5	1	1.5	2	2.5	3
$f_Y(y)$	0.00	0.19	0.33	0.42	0.44	0.42	0.33

mean=1.75 median=1.79

c. Find and graph F_Y (y), the corresponding CDF, from $0 \le y \le 3$ (plot every 0.5).

$$F_Y(y) = \int_0^y f_Y(t)dt = \frac{1}{9}\int_0^y t(4-t)dt = \frac{1}{9}\left(2t^2 - \frac{t^3}{3}\right)\Bigg|_{t=0}^{y} = \frac{1}{9}\left(2y^2 - \frac{y^3}{3}\right) = \frac{y^2}{27}(6-y)$$



y	0	0.5	1	1.5	2	2.5	3
$F_Y(y)$	0.00	0.05	0.19	0.38	0.59	0.81	1.00

median=1.79

d. Find $P(|Y-1| < 0.5)$. Graph the inequality on your plot from part b) and shade the area representing the probability.

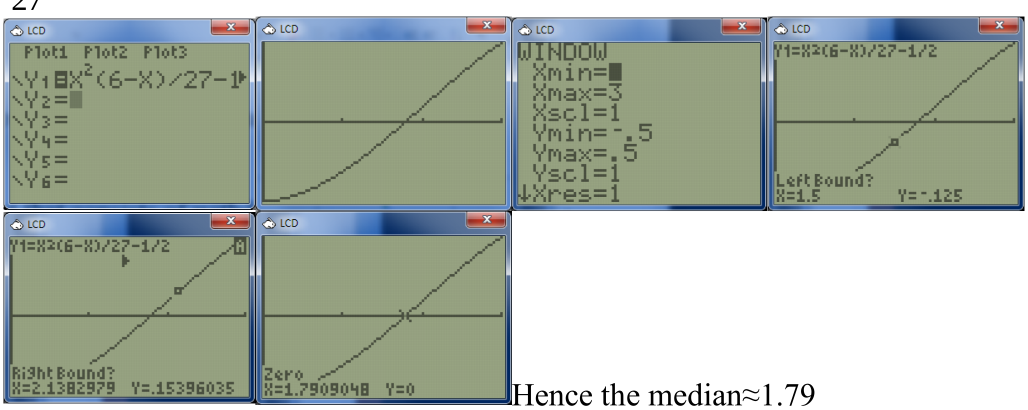
$$P(|Y-1| < 0.5) = F_Y(1.5) - F_Y(0.5) = \frac{(1.5)^2}{27}(6-1.5) - \frac{(0.5)^2}{27}(6-0.5) \approx 0.32$$

e. Find the expectation and label on your plot from part b)

$$E(Y) = \int_0^3 y\, f_Y(y)dy = \int_0^3 y^2(4-y)dy = \frac{1}{9}\left(\frac{4y^3}{3} - \frac{y^4}{4}\right)\Bigg|_{y=0}^{3} = \frac{1}{9}\left(\frac{4*3^3}{3} - \frac{3^4}{4}\right) = \frac{1}{9}\left(\frac{36}{1} - \frac{81}{4}\right) = \frac{7}{4}$$

f. Find the median and label on both graphs (the first with a vertical line and the second with a horizontal line and a vertical line).
To find the median, set F_Y (y)=1/2 and solve for y:

$$\frac{y^2}{27}(6-y) = 1/2 \Rightarrow \text{(use numeric equation or root finder on calculator)}$$



Hence the median≈1.79

g. Compare relative positions of the median and the expectation. Explain how comparison relates to any skewing. **The mean is to the left of the median, therefore the skewing is to the left.**

3. (40 pts) Consider an experiment that consists of withdrawing a ball from the box, NOT replacing it, and withdrawing a second ball. There are 2 red and 4 green balls in the box. An outcome will be a coordinate of 2 colors, say (r,r), which can be shortened to rr.

a. What is the sample space of this experiment? Is this a random variable? Why or why not?
$\{rg, gr, gg, rr\}$ No, outcomes are not numbers.

b. To get a random variable W, we count the number of red balls selected. Find the possible outcomes and express their probabilities as quotients of binomial coefficients. Write W as a table with row 1 as the outcomes and row 2 the corresponding probabilities.

w	0	1	2
$P(w)$	$\frac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}}$	$\frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}}$	$\frac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}}$

c. Evaluate the binomial coefficients, leaving as fractions. Write W as a matrix with row 1 as the outcomes and row 2 the corresponding probabilities. Rewrite table representing W.

w	0	1	2
$P(w)$	$\frac{1*6}{15}=\frac{6}{15}$	$\frac{2*4}{15}=\frac{8}{15}$	$\frac{1*1}{15}=\frac{1}{15}$

d. Use the 2 row table of c to find the expectation μ=E(W).
Expectation is dot product of the 2 rows above thought of as vectors: 0*6/15+1*8/15+2*1/15=10/15=2/3

e. Find the 2 row table for W^2 and use it find $\mu_2=E(W^2)$.

w^2	0	1	4
$P(w)$	$\frac{6}{15}$	$\frac{8}{15}$	$\frac{1}{15}$

$\mu_2=E(W^2)=8/15+4/15=12/15=4/5$

f. Use the results from d and e to find V(W). $V(W) = \mu_2 - \mu^2 = 4/5 - (2/3)^2 = 36/45 - 20/45 = 16/45$

g. If X is an RV indicating whether the first ball is red or not (1, 0 respectively) and Y is whether the second ball is red or not (1, 0 respectively). Find E(X) and E(Y) and show that their sum corresponds to your answer in d.

X:

x^2	0	1
x	0	1
$P(x)$	$\frac{2}{3}$	$\frac{1}{3}$

& Y:

y^2	0	1
y	0	1
$P(y)$	$\frac{2}{3}$	$\frac{1}{3}$

so E(X)=E(Y)=1/3 and so E(X+Y)= E(X)+E(Y)=1/3+1/3=2/3

h. Find V(X) and V(Y) for X and Y described in f. and show that their sum DOES NOT correspond to your answer in d. Why not? Try to reason why it is more or less.
$V(X)=V(Y)=E(X^2)-E(X)^2=1/3-(1/3)^2=(3-1)/9=2/9$
Note that $16/45=V(X+Y)\neq V(X)+V(Y) = 2/9 + 2/9=4/9=20/45$
X and Y are dependent so V(X+Y)=V(X)+V(Y) does not hold. Basically, the variability in the result gets reduced because of the dependence.