Algorithms

Chapter 3

With Question/Answer Animations

Chapter Summary

- Algorithms
  - Example Algorithms
  - Algorithmic Paradigms
- Growth of Functions
  - Big-*O* and other Notation
- Complexity of Algorithms

Complexity of Algorithms

Section 3.3

Section Summary

- Time Complexity
- Worst-Case Complexity
- Algorithmic Paradigms
- Understanding the Complexity of Algorithms

The Complexity of Algorithms

- Given problem, algorithm and input of a particular size, how efficient is algorithm for solving problem?
- In particular
  1. how much time does algorithm use?
  2. how much computer memory?
- When we analyze
  1. time, we are finding *time complexity;*
  2. computer memory, we find its *space complexity*.

Time Complexity

- Will be our Focus for both 2440 and 2540.

  (You may find space complexity treated in other courses.)
- Our "time" measure is #"operations"
- We often use big-*O* (big-Θ) notation.
- Using this analysis, we can
  1. determine how practical it is to use algorithm with input of a particular size.
  2. compare efficiency of different algorithms for solving same problem.
- We ignore implementation details, e.g.,
  - data structures
  - hardware and software platforms

Time Complexity (cont.)

- "operation" is some specific part of the procedure. In our course, it will be either a
  - comparison or
  - arithmetic operation (addition, multiplication, etc.);
- We often ignore “house keeping” aspects, like memory swaps and assignments.
- If we determine #"operations" and time needed for 1 "operation"
  - then we can estimate actual time a computer needs provided that this "operation" accounts for most of execution time.
- We focus on *worst-case* of an algorithm which provides an upper bound on #operations needed.
- On occasion we find *average-case* which is the average #operations used over all inputs of particular size.

Example: max, a precise approach

(finds the maximum element in a finite sequence)

Find time complexity of *max*: count #comparisons

```
procedure max(a1, a2, …., an: integers)
    max := a1
    for i := 2 to n
        if max < ai then max := ai
    return max{max is the largest element}
```

Solution: NOTE: there are 2 places for comparisons:

- within the loop: $max < a_i$
  (comparison is made $n - 1$ times, once for pass of loop)
- at beginning of each pass: we see if loop variable i is $\leq n$.
  (comparison is made n times, i:=2 to n+1)
- Hence, exactly $(n - 1) + n = 2n - 1$ comparisons are made.

Thus, the time complexity of the algorithm is $\Theta(n)$.

Example: max, a more common approach

(finds the maximum element in a finite sequence)

Find time complexity of *max*: count #comparisons

```
procedure max(a1, a2, ...., an: integers)
    max := a1
    for i := 2 to n
        if max < ai then max := ai
    return max{max is the largest element}
```

A more commonly used approach is to say that

- the comparison that is part of the hidden mechanisms of the loop is part of "housekeeping" and hence can be ignored.

For this problem, while the #comparisons

- then goes down to $n - 1$
- we still get the same $\Theta(n)$ complexity result

Example: Linear Search

Determine **worst-case** complexity of linear search.

procedure *linear search*(x: integer, $a_1, a_2, \ldots, a_n$: distinct integers)
$i := 1$
while ($i \leq n$ and $x \neq a_i$)
    $i := i + 1$
if $i \leq n$ **then** *location* := i
else *location* := 0
return *location*{*location* is the subscript of the term that equals x, or is 0 if x is not found}

Solution: Count #comparisons.

- If x is in list, then loop will exit early $\Rightarrow \exists$fewer comparisons
- Thus, we assume that x is **not** in the list.
- For each pass of loop, 2 comparisons are made; $i \leq n$ and $x \neq a_i$.
- To exit loop, comparison $n + 1 \leq n$ is made.
- After the loop, one more $n + 1 \leq n$ comparison is made.

So in worst case, $2n + 2$ comparisons are made.

$\therefore$ complexity is $\Theta(n)$.

Example: Linear Search (cont.)

Determine **average-case** complexity of linear search.

procedure *linear search*(x: integer, $a_1, a_2, \ldots, a_n$: distinct integers)
$i := 1$
while ($i \leq n$ and $x \neq a_i$)
    $i := i + 1$
if $i \leq n$ **then** *location* := i
else *location* := 0
return *location*{*location* is the subscript of the term that equals x, or is 0 if x is not found}

Solution: This time we assume that x is in list, say in j^{th} location, i.e., $x = a_j$

- For each pass of loop, 2 comparisons are made: $i \leq n$ and $x \neq a_i$.
- To exit loop, comparison $j \leq n$ (T) is made as well as $x \neq a_j$ (F).
- After the loop, one more $j \leq n$ comparison is made.

So $2j + 1$ comparisons are made. j could be any of 1,.., n and hence,

$$\text{ave} = \frac{1}{n}\sum_{j=1}^{n} 2j + 1 = \frac{1}{n}\left[2\left(\sum_{j=1}^{n} j\right) + n\right] = \frac{1}{n}\left[2\frac{n(n+1)}{2} + n\right] = n + 2$$

$\therefore$ average-case complexity is $\Theta(n)$. (For those don't like Σ, see next slide.)

Example: Linear Search (cont.)

- For those who do not like to work with Σ:

$$\frac{3+5+7+\ldots+(2n+1)}{n} = \frac{2(1+2+3+\ldots+n)+n}{n} = \frac{2[\frac{n(n+1)}{2}]}{n} + 1 = n+2$$

- From now on, "complexity" will mean

"worst-case time complexity"

unless otherwise stated.

Example: Binary Search

Find complexity of binary search in terms of #comparisons.

procedure binary search(x: integer, $a_1, a_2, \ldots, a_n$: increasing integers)
 $i := 1$ {i is the left endpoint of interval}
 $j := n$ {j is right endpoint of interval}
 while $i < j$
  $m := \lfloor (i + j)/2 \rfloor$
  if $x > a_m$ then $i := m + 1$
  else $j := m$
 if $x = a_i$ **then** *location* $:= i$
 else *location* $:= 0$
 return *location*{location is the subscript i of the term a_i equal to x, or 0 if x is not found}

Solution: Assume for moment that $n = 2^k$ elements. Note that $k = \log n$.

- Two comparisons are made for each pass: $i < j$, and $x > a_m$.
- Before 1st iteration, size of list is 2^k, then 2^{k-1} , 2^{k-2} & so on until size of list is $2^1 = 2$.
- Loop exits when |list| is $2^0 = 1$ & then x is compared with single remaining element.
- Hence, $2k + 2 = 2 \log n + 2$ comparisons are made.
- For non-powers of 2, #comparisons is $2 \lceil \log n \rceil + 2$

$\therefore$ complexity is $\Theta(\log n)$, which is better than linear search.

Example: Bubble Sort

Find complexity of bubble sort in terms of #comparisons

procedure *bubblesort*(a_1,...,a_n: real numbers with $n \geq 2$)
  for $i := 1$ to $n-1$
    for $j := 1$ to $n-i$
      if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}
{a_1,..., a_n is now in increasing order}

Solution: A sequence of $n-1$ passes is made through list.
On i^{th} pass $n-i$ comparisons are made.
Hence #comparisons is

$$(n-1)+(n-2)+\ldots+2+1 = \frac{n(n-1)}{2}$$

$\therefore$ complexity of bubble sort is $\Theta(n^2)$.

Example: Insertion Sort

Find complexity of insertion sort in terms of #comparisons

Solution: If we study # passes for 2 serially constructed while and for-loops inside the outer for-loop, we see that #comparisons is exactly j, the index of the outer loop.
Hence #comparisons in total is:

$$2 + 3 + \cdots + n = \frac{n(n-1)}{2} - 1$$

$\therefore$ complexity is $\Theta(n^2)$.

procedure *insertion sort*(a_1,...,a_n: real numbers with $n \geq 2$)
  for $j := 2$ to n
    $i := 1$
    while $a_j > a_i$
      $i := i + 1$
    $m := a_j$
    for $k := 0$ to $j - i - 1$
      $a_{j-k} := a_{j-k-1}$
    $a_i := m$

Matrix Multiplication Algorithm

- The definition for matrix multiplication can be expressed as an algorithm; $\mathbf{C} = \mathbf{A}\,\mathbf{B}$ where $\mathbf{C}$ is an $m \times n$ matrix that is the product of the $m \times k$ matrix $\mathbf{A}$ and the $k \times n$ matrix $\mathbf{B}$.
- This algorithm carries out matrix multiplication based on its definition.

procedure *matrix multiplication*($\mathbf{A}$,$\mathbf{B}$: matrices)
  for $i := 1$ to m
    for $j := 1$ to n
      $c_{ij} := 0$
      for $q := 1$ to k
        $c_{ij} := c_{ij} + a_{iq}\, b_{qj}$
return $\mathbf{C}$ {$\mathbf{C} = [c_{ij}]$ is the product of $\mathbf{A}$ and $\mathbf{B}$}

$\mathbf{A} = [a_{ij}]$ is a $m \times k$ matrix
$\mathbf{B} = [b_{ij}]$ is a $k \times n$ matrix

Example: Matrix Multiplication

Find complexity in terms of #arithmetic operations

Solution: Product of 2 $n \times n$ matrices is itself an $n \times n$ matrix and so has n^2 entries.

Using procedure on previous slide:

- finding an entry requires n mults & n additions

[doing by hand uses one fewer addition:

$(i,j)^{\text{th}}$ entry is the dot product of i^{th} row and j^{th} column].

Hence, n^3 multiplications & n^3 additions are used.

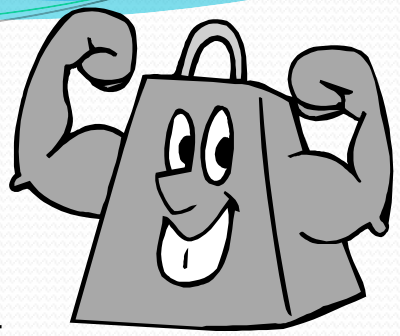
$\therefore$ complexity is $O(n^3)$.

Algorithmic Paradigms

An *algorithmic paradigm* is general approach based on particular concept for constructing algorithms to solve variety of problems.

- Greedy algorithms were introduced in Section 3.1.
- We discuss brute-force algorithms in this section.
- Elsewhere in text you can find:
  - probabilistic algorithms (Chapter 7)
  - divide-and-conquer algorithms (Chapter 8)
  - dynamic programming (Chapter 8)
  - backtracking (Chapter 11)
- ∃many other paradigms that you may see in other courses.

Brute-Force Paradigm



A *brute-force* algorithm is solved in the most straightforward manner, without taking advantage of any ideas that can make the algorithm more efficient.

- Brute-force algorithms we have previously seen:
  - sequential search
  - bubble and insertion sort

Example: Closest Pair of Points

Construct algorithm for finding closest pair of points in a set of *n* points & find complexity in terms of # operations.

Solution: Recall that distance betwn pts (x_i, y_i) & (x_j, y_j) is

$$\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

Our brute-force algorithm computes distance between all pairs of points & picks pair with smallest distance.

Note: In our procedure, we do not compute the square root, since the square of the distance between two points is smallest when the distance is smallest.

Procedure and estimate →

Closest Pair of Points (cont.)

Algorithm for finding the closest pair in a set of n points.

procedure *closest pair*$((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$: x_i, y_i real numbers$)$
min = ∞
  for $i := 1$ to n
    for $j := 1$ to i
      if $(x_j - x_i)^2 + (y_j - y_i)^2 < min$
        then min := $(x_j - x_i)^2 + (y_j - y_i)^2$
          closest pair := $(x_i, y_i), (x_j, y_j)$
return *closest pair*

- The algorithm loops through $n(n-1)/2$ pairs of points, computes the value $(x_j - x_i)^2 + (y_j - y_i)^2$ and compares it with the minimum, etc.
- So our algorithm uses $\Theta(n^2)$ arithmetic and comparison operations.
- For algorithm with $O(\log n)$ worst-case complexity, see Section 8.3.

Terminology for Complexity

TABLE 1 **Commonly Used Terminology for the Complexity of Algorithms.**

Complexity	*Terminology*
$\Theta(1)$	Constant complexity
$\Theta(\log n)$	Logarithmic complexity
$\Theta(n)$	Linear complexity
$\Theta(n \log n)$	Linearithmic complexity
$\Theta(n^b)$	Polynomial complexity
$\Theta(b^n)$, where $b > 1$	Exponential complexity
$\Theta(n!)$	Factorial complexity

Comparison table for Complexity

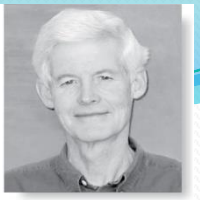
Problem Size	**Computer Time Used by Algorithms.**				
	Bit Operations Used				
n	$\log n$	n	$n \log n$	n^2	2^n
10	3×10^{-11} s	10^{-10} s	3×10^{-10} s	10^{-9} s	10^{-8} s
10^2	7×10^{-11} s	10^{-9} s	7×10^{-9} s	10^{-7} s	4×10^{11} yr
10^3	1.0×10^{-10} s	10^{-8} s	1×10^{-7} s	10^{-5} s	*
10^4	1.3×10^{-10} s	10^{-7} s	1×10^{-6} s	10^{-3} s	*
10^5	1.7×10^{-10} s	10^{-6} s	2×10^{-5} s	0.1 s	*
10^6	2×10^{-10} s	10^{-5} s	2×10^{-4} s	0.17 min	*

- Times of more than 10^{100} years are indicated with an *.
- We assume each operation takes 10^{-11}s = 0.01 nano s = 10 pico s
- 10^{11} yr is 100 billion yrs (BY). In contrast age of universe is 13.8 BY

Complexity of Problems

- *Tractable*: ∃polynomial time alg. to solve problem (*Class P*).
- *Intractable*: ∄polynomial time alg. to solve problem
- *Unsolvable*: ∄alg. to solve problem, e.g., halting problem.
- *Class NP*: Solution can be checked in polynomial time. But no polynomial time alg. has been found for finding solution.
- *NP Complete Class*: If you find polynomial time alg. for one member of class, it can be used to solve all problems in class.

P Versus NP Problem



Stephen Cook
(Born 1939)

P versus NP problem asks whether class P ≠ NP?

- I.e., do ∃problems whose solutions can be *checked* in poly. time, but can not be *solved* in poly. time?
- If polynomial time algorithm for *any* problem in the NP complete class were found, then that algorithm could be used to obtain a polynomial time algorithm for *every* problem in the NP complete class.
  - Satisfiability (in Section 1.3) is an NP complete problem.
- It is generally believed that P≠NP since no one has been able to find a polynomial time algorithm for any problem in the NP complete class.
- P versus NP is the most famous **unsolved** problems in theoretical CS.
- **The Clay Math Institute has $1,000,000 prize for solution!**