Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- Matrices
  - Matrix Arithmetic

Sets

Section 2.1

Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an **unordered** collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Repeated elements, use of elipses

- $S = \{a,b,c,d\}$
- Order not important

  $$S = \{a,b,c,d\} = \{b,c,a,d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

  $$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $$S = \{a,b,c,d, \ldots\ldots,z\}$$

Examples

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$
- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$
- Set of all positive integers less than 100:

  $S = \{1,2,3,........,99\}$
- Set of all integers less than 0:

  $S = \{...., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^+$ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

R \ **Q** = set of irrational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \in \mathbf{Z}^+ \text{ and } x < 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid P(x)\}$$

  - Example: $S = \{x \mid \text{Prime}(x)\}$

- Positive rational numbers:

  $$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$
$$[a,b) = \{x \mid a \le x < b\}$$
$$(a,b] = \{x \mid a < x \le b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

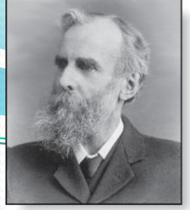
Interval Notation

$[a,b] = \{x \mid a \leq x \leq b\}$ closed!

$[a,b) = \{x \mid a \leq x < b\}$ half-open (or half-closed!)

$(a,b] = \{x \mid a < x \leq b\}$ ditto

$(a,b) = \{x \mid a < x < b\}$ open!



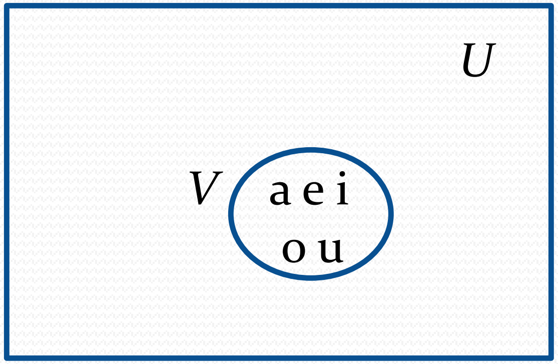
John Venn (1834-1923)
Cambridge, UK

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements.

(Symbolized ∅, but {} is also used.)

Venn Diagram



Russell's Paradox

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

- Let *S* be the set of all sets which are not members of themselves.
- A paradox results from trying to answer the question "Is *S* a member of itself?"
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\}, a, \{b, c\}\}$

  $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

- The empty set is different from a set **containing** the empty set.

  $\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- If A and B are sets,

then A and B are equal if and only if

$$\forall x (x \in A \leftrightarrow x \in B)$$

.

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3,5,1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S.
  2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

Proper Subsets

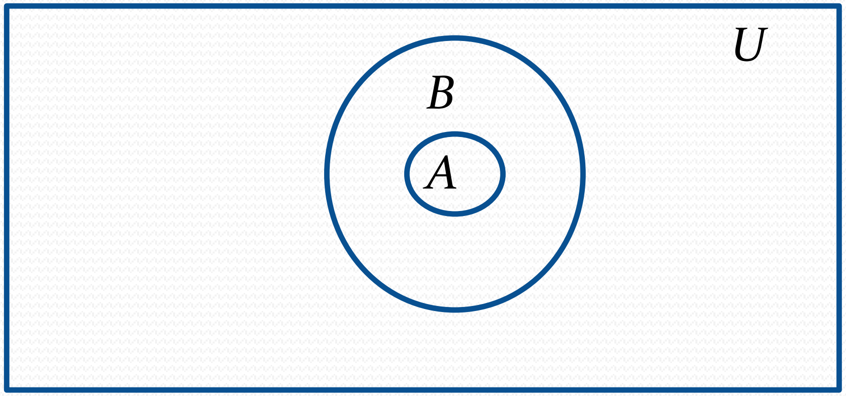
Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$.

If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$$

is true.

Venn Diagram



Exercise: Give an example of proper subset of Z.

Set Cardinality

Definition: If ∃n ∈ **N** distinct elements in *S*, we say that *S* is *finite*. Otherwise it is *infinite*. The *cardinality* of set *A is* denoted by |*A*|. If A is a finite, then |*A*| is this same n.

Examples:

1. |ø| = 0
2. Let S be the letters of the English alphabet. Then |*S*| = 26
3. |{1,2,3}| = 3
4. |{ø}| = 1
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example: If $A = \{a,b\}$ then

$$\mathcal{P}(A) = \{ø, \{a\},\{b\},\{a,b\}\}$$

- If a set has n elements, then the cardinality of the power set is 2^n.

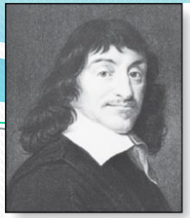
(Chapters 5 and 6 have several different ways to show this.)

Tuples

The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its 1st element and a_2 as its 2nd element and so on until a_n as its last element.

- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product



René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

Example: $A \times B = \{(a, b) | a \in A \wedge b \in B\}$

$A = \{a,b\}, \quad B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B. (Relations are covered in depth in Chapter 9.)

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where $a_i \in A_i$ for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: Find $A \times B \times C$, if $A = \{0,1\}$, $B = \{1,2\}$, $C = \{0,1,2\}$

Solution:

$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "|*x*| = 1" is the set {-1,1}