The Foundations: Logic and Proofs

Chapter 1, Part II: Predicate Logic

With Question/Answer Animations

Chapter Summary

- Propositional Logic
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- Predicate Logic
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
- Proofs
  - Rules of Inference
  - Proof Methods
  - Proof Strategy

Summary

- Predicate Logic (First-Order Logic (FOL), Predicate Calculus)
  - The Language of Quantifiers
  - Logical Equivalences
  - Nested Quantifiers
  - Translation from Predicate Logic to English
  - Translation from English to Predicate Logic

Nested Quantifiers

Section 1.4

Section Summary

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an additive inverse" is

  $$\forall x \exists y \, (x + y = 0)$$

  where the domains of x and y are the real numbers.

- We can also use nested propositional functions:

$\forall x \exists y \, (x + y = 0)$ can be viewed as

$\forall x \, Q(x)$ where $Q(x)$ is $\exists y \, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Nested Quantification & loops

Example 1: $\forall x \, \forall y \, P(x,y)$

- loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false:
        - terminate both the outer and inner loops;
        - $\forall x \, \forall y \, P(x,y)$ has been determined to be false.
    - If there is no such pair and the outer loop ends
        - $\forall x \, \forall y \, P(x,y)$ has been determined to be true.
- If one (or both) of the domains for the variables is infinite, then this process will not terminate under certain circumstances, in particular if the proposition is true.

Nested Quantification & loops

Example 2: $\forall x \exists y\ P(x,y)$

loop through the values of x:

- For each x, loop through the values for y:
  - The inner loop ends when a pair x and y is found such that P(x, y) is true. Proceed to next value of x.
  - If the inner loop terminates and no such y is found
    - Terminate the outer loop
    - $\forall x \exists y\, P(x,y)$ has been shown to be false.
- if the outer loop ends after stepping through each x:
  - $\forall x \exists y\, P(x,y)$ has been shown to be true.

As before, an infinite domain (or 2) may mean non-termination.

Order of Quantifiers

Quantifiers of the same type commute.

Quantifiers of opposite types do not (in general).

Examples:

1. Let $P(x,y) =$ "$x + y = y + x$", $U =$ real numbers.

   Then $\forall x\, \forall y\, P(x,y) \equiv \forall y\, \forall x\, P(x,y)$

   (and hence we may say $\forall x, y$ *)*

2. Let $Q(x,y) =$ "$x + y = 0$", $U =$ real numbers. Then
   - $\forall x\, \exists y\, Q(x,y)$ is true; *(why?)*
   - $\exists y\, \forall x\, Q(x,y)$ is false. (why?)

Order of Quantifiers: exercise A

Let $P(x,y)$ = "$x \cdot y = 0$" and U = real numbers,

What is the truth value of the following:

1. $\forall x \forall y\, P(x,y)$

   Answer: False

2. $\forall x\, \exists y\, P(x,y)$

   Answer: True

3. $\exists x \forall y\, P(x,y)$

   Answer: True

4. $\exists x\, \exists\, y\, P(x,y)$

   Answer: True

Order of Quantifiers: exercise B

Let $P(x,y)$ = "$x \cdot y = 1$" and U = real numbers,

What is the truth value of the following:

1. $\forall x \forall y\, P(x,y)$

   Answer: False

2. $\forall x\, \exists y\, P(x,y)$

   Answer: False

3. $\exists x \forall y\, P(x,y)$

   Answer: False

4. $\exists x\, \exists\, y\, P(x,y)$

   Answer: True

Can you restrict the domain by removing just a single value, so that one of the "false" 's becomes true?

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Translating Nested Quantifiers into English

Let C(x) = "*x* has a computer," *F*(*x*,*y*) = "*x* and *y* are friends," and domain = "all students in your school". Translate

Example 1: $\forall x\ (C(x) \lor \exists y\ (C(y) \land F(x, y)))$

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: $\exists x\ \forall y\ \forall z\ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$

Solution: There is a student none of whose friends are also friends with each other.

Math Statements & Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite to make implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y:

   "For all positive integers x and y, $x + y$ is positive."

3. Specify the domain: **integers**.
4. The result is: $\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$

Translating English into Logical Expression

Example: "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ = "w has taken f", $Q(f,a)$ = "f is a flight on a."
2.

variable	w	f	a
domain	women	flights	airlines

3. Expression: $\exists w\ \forall a\ \exists f\ (P(w,f) \land Q(f,a))$

More Translation from English

Exercise: Choose obvious predicates and express in predicate logic.

1: "Brothers are siblings."

Solution: $\forall x\ \forall y\ (B(x, y) \rightarrow S(x, y))$

2: "Siblinghood is symmetric."

Solution: $\forall x\ \forall y\ (S(x, y) \rightarrow S(y, x))$

3: "Everybody loves somebody."

Solution: $\forall x\ \exists y\ L(x, y)$

4: "There is someone who is loved by everyone."

Solution: $\exists y\ \forall x\ L(x, y)$

5: "There is someone who loves someone."

Solution: $\exists x\ \exists y\ L(x, y)$

6: "Everyone loves him/herself"

Solution: $\forall x\ L(x, x)$

Negating Nested Quantifiers

Exercise: Recall logical expression developed 2 slides back:

$$\exists w \forall a \exists f \, (P(w,f) \land Q(f,a))$$

1: Write expression for its negation, i.e. "There does **not** exist a woman who has taken a flight on every airline in the world."

$$\neg \exists w \forall a \exists f \, (P(w,f) \land Q(f,a))$$

2: Use De Morgan's Laws to move negation inwards:

i. $\forall w \neg \forall a \exists f \, (P(w,f) \land Q(f,a))$
ii. $\forall w \exists a \neg \exists f \, (P(w,f) \land Q(f,a))$
iii. $\forall w \exists a \forall f \neg (P(w,f) \land Q(f,a))$
iv. $\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$

3: Translate the result back into English:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

Test your understanding Question 1

Can you switch the order of quantifiers?

- Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.

- Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

Solution: No! The left and the right side may have different truth values for some propositional functions for *P*.

Try "x + y = 0" for *P(x,y)* with *U* being the integers. The order in which the values of *x* and *y* are picked does matter.

Test your understanding Question 2

Can you distribute quantifiers over logical connectives?

- Is this a valid equivalence? $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.

- Is this a valid equivalence? $\forall x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

Solution: No! Let P(x) =“x is a fish”, Q(x) = “x has scales” with domain all animals.

Then the left side is false, because there are some fish that do not have scales.

But the right side is true since not all animals are fish.