Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

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Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- ~~Matrices~~
  - ~~Matrix Arithmetic~~

Sequences and Summations

Section 2.4

Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences

List definition & introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, ...
- Sequences arise in mathematics, computer science, and many other disciplines, e.g., botany & music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Function definition & notation

A *sequence* is a function from a subset of the integers (usually either {0, 1, 2, ...} or {1, 2, 3, ...}) to a set S.

- In place of $a(n)$, use a_n to denote the image of $n \in \mathbf{Z}$.
- We call a_n a *term* of the sequence.

Notation options

Example: Consider the *harmonic* sequence

$$\{a_n\} = \{a_1, a_2, a_3, \ldots\}$$

where

$$a_n = \frac{1}{n} \qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$$

Geometric Progression

A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

with *initial term a* and *common ratio r, both real #'s.*

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

An *arithmetic progression* is a sequence of the form:

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

with *initial term a* and *common difference d, both* real #'s.

Examples:

1. Let *a* = −1 and *d* = 4:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let *a* = 7 and *d* = −3:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let *a* = 1 and d = 2:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

A *string* is a finite sequence of characters from a finite set (e.g., the alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

A *recurrence relation* is an equation that expresses a_n in terms of one or more of a_0, a_1, …, a_{n-1}

- The relation is valid for $\forall\ n \geq n_0$ where $n_0 \in Z^+$.
- *Initial conditions* specify the terms a_n for $n < n_0$.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relation Questions

Example 1: Given recurrence relation $a_n = a_{n-1} + 3$ for $n_o = 1$ and $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_o = 2$ is the single initial condition.]

Solution: We see from the recurrence relation that

$a_1 = a_o + 3 = 2 + 3 = 5$

$a_2 = 5 + 3 = 8$

$a_3 = 8 + 3 = 11$ and so on

Does anyone recognize this sequence?

Recurrence Relations Questions

Example 2: Let $a_n = a_{n-1} - a_{n-2}$, $n_o = 2$ and $a_0 = 3$, $a_1 = 5$. What are a_2, a_3, a_4, a_5 and a_6?

[Here the initial conditions is the pair $a_o = 3$, $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_o = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

$$a_4 = a_3 - a_2 = -3 - 2 = -5$$

$$a_5 = a_4 - a_3 = -5 - (-3) = -2$$

$$a_6 = a_5 - a_4 = -2 - (-5) = 3$$

What is overall shape?

Quadratic (parabola)? Periodic (like a sin curve)?

Fibonacci Sequence

$f_n = f_{n-1} + f_{n-2}$, $n_o = 2$ and $f_0 = 0$, $f_1 = 1$

Example: Find $f_2, f_3, \ldots, f_6$.

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

f_n is #pairs of rabbits after n months, where a new pair does not breed its 1st month but produces 1 new pair of offspring each month thereafter.

Solving Recurrence Relations

Often, we *solve* (find a closed formula) for the n^{th} term:

- In this chapter, we use *methods of iteration* where we guess the formula.
- The guess can be proved correct by induction (Chap 5).
- Chapter 8 has various advanced methods for solving.

Forward Iterative Solution

$a_n = a_{n-1} + 3,\ n_o = 2, a_1 = 2.$

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 2 \cdot 3$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$a_n = 2 + 3(n - 1)$$

Backward Iterative Solution

$a_n = a_{n\text{-}1} + 3,\ n_o = 2, a_1 = 2.$

$$a_n = a_{n\text{-}1} + 3$$

$$= (a_{n\text{-}2} + 3) + 3 = a_{n\text{-}2} + 3 \cdot 2$$

$$= (a_{n\text{-}3} + 3) + 3 \cdot 2 = a_{n\text{-}3} + 3 \cdot 3$$

$$\vdots$$

$$= a_2 + 3(n - 2)$$

$$= (a_1 + 3) + 3(n - 2)$$

$$= 2 + 3(n - 1)$$

Financial Application

Example: Suppose that a person deposits $10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\ P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}\,,\ P_0 = 10{,}000$$

Solution: Forward Substitution

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2 P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3 P_0$$

$$\vdots$$

$$P_n = (1.11)P_{n-1} = (1.11)^n \mathrm{P}_0$$

$$= (1.11)^n\, 10{,}000$$

$$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$$

Identifying Integer Sequences

Given a few terms of a sequence, try to identify sequence.

Conjecture formula, recurrence relation, or other rule.

- Some questions to ask, tools to use:
  - Are there repeated terms of the same value?
  - Can you obtain term from previous term adding (arithmetic) or multiplying (geometric)by an amount?
  - Can you obtain a term by combining the previous terms?
  - Are their cycles among the terms?
  - (online encyclopedia of integer sequences)

Questions on Integer Sequences

Example 1: 1, ½, ¼, 1/8, 1/16,…

Solution: Denominators are powers of 2;

geometric progression with $a = 1$ and $r = ½$: $a_n = 1/2^n$

Example 2: 1,3,5,7,9,…

Solution: Term obtained by adding 2 to previous term;

arithmetic progression with $a = 1$, $d = 2$: $a_n = 2n + 1$

Example 3: 1, -1, 1, -1,1, …

Solution: Terms alternate between 1 and -1;

geometric progression with $a = 1$, $r = -1$: $a_n = (-1)^n$

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Finding Sequences

Example: Find simple formula for a_n if 1st 10 terms are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: Ratio of each term to the previous ≈ 3.

So now compare with the sequence 3^n .

We notice that the nth term is 2 less than the corresponding power of 3 so

$$a_n = 3^n - 2.$$

Integer Sequence Puzzles

Identify these 3 sequences from the OEIS site:

1. 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, ...
   Think of alternatively adding & multiplying by successive #s.
2. 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
   one, two, three, four, five, six, seven, eight, nine, ten, eleven, ...
   and think like a Roman! (c,d,i,l,m,v,x)
3. 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
   Think of the English names for numbers, and whether or not they have the letter 'e.'

The answers and many more can be found at

http://oeis.org/Spuzzle.html

Summations

Sum of terms $a_m, a_{m+1}, \ldots, a_n$ from sequence $\{a_n\}$

is $a_m + a_{m+1} + \cdots + a_n$

$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \sum_{m \le j \le n} a_j$$

Display Inline Subscript

- Variable j is the *index of summation*.

It runs through all the integers starting with its *lower limit* m and ending with its *upper limit* n.

Summations

More generally, for set *S*, the sum of all its elements is:

$$\sum_{j \in S} a_j$$

Examples:

- Sum of powers 0,1,…,n (special case of geometric)

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_0^n r^j$$

- Harmonic: it *diverges* (gets arbitrarily large)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_1^\infty \frac{1}{i}$$

- Finite set:

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

- Product of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$ is
$$a_m \times a_{m+1} \times \cdots \times a_n$$
- Some possible notations:

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

display inline subscript

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

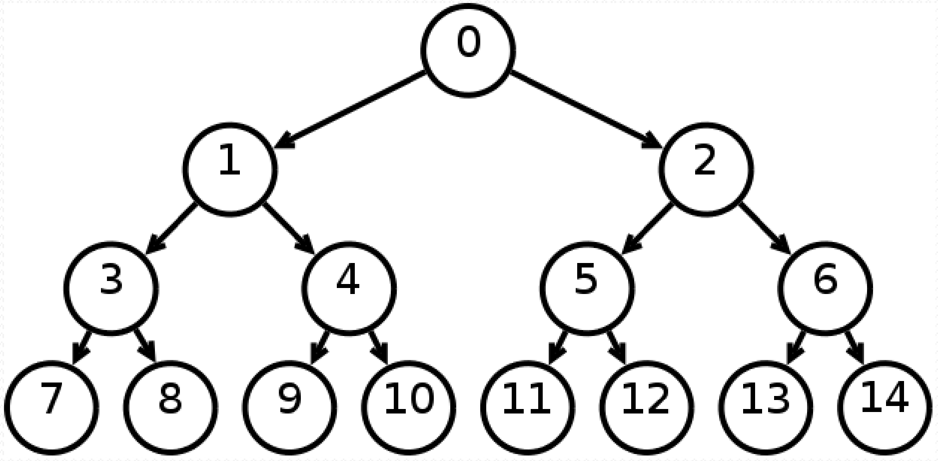
Example: A full binary tree of depth 3 has

1+2+4+8 = $\frac{1*16-1}{2-1} = 15$

nodes (pictured on right) and one of depth 10 has

1+2+…+1024 = $\frac{1*2048-1}{2-1}$

= 2047 nodes



Geometric Series: proof (r ≠1)

$$rS_n = r\sum_{j=0}^{n} ar^j$$ First multiply both sides of the equality by r

$$= \sum_{j=0}^{n} ar^{j+1}$$ Move factor of r into the summation

$$= \sum_{k=1}^{n+1} ar^k$$ Shifting the index of summation with $k = j + 1$.

$$= \left(\sum_{k=0}^{n} ar^k\right) + (ar^{n+1} - a)$$ Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + (ar^{n+1} - a)$$ Substituting S_n for summation formula

$$rS_n = S_n + (ar^{n+1} - a) \qquad S_n = \frac{ar^{n+1} - a}{r - 1}$$ Solving for S_n

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)