Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- ~~Matrices~~

Functions

Section 2.3

Section Summary

- Definition of a Function.
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (not in syllabus)

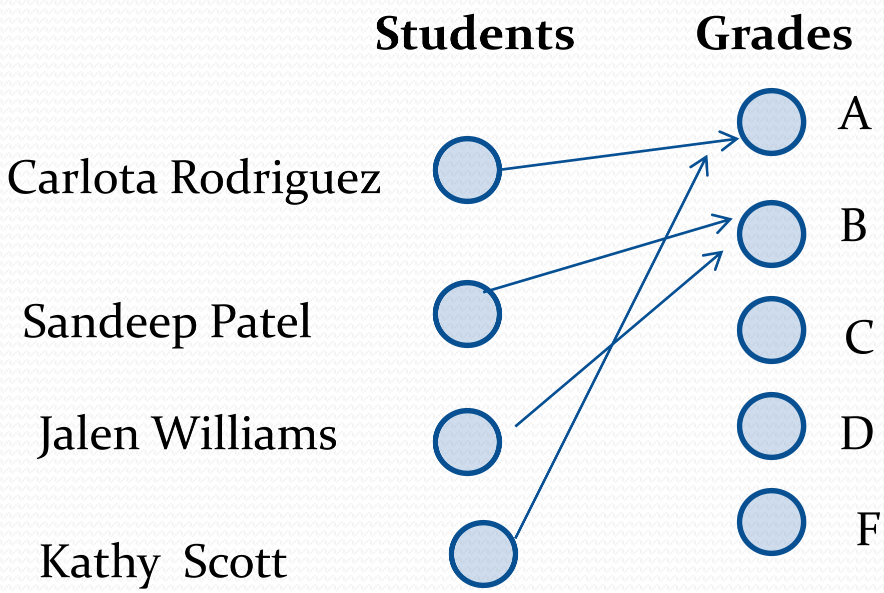
Functions

Let A and B be nonempty sets.

A *function or mapping* f from A to B, denoted $f : A \to B$, is an assignment of each $a \in A$ to **exactly** one $b \in B$.

We write $f(a) = b$ **or** $f : a \mapsto b$.

- Each student is *mapped* to a particular grade.

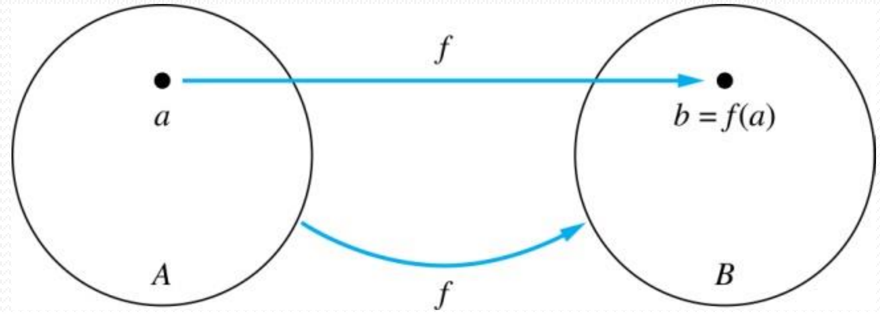


Functions

- A function $f : A \rightarrow B$ can also be defined as a relation (a subset of $A \times B$) where no two elements of the relation have the same first element.
- In other words, there is one and only one ordered pair (a, b) for every element $a \in A$:

$$\forall x [x \in A \rightarrow \exists ! \, y [y \in B \wedge (x, y) \in f]]$$

Functions



Given a function $f: A \to B$

- A is the *domain*.
- B is the *codomain*.
- If $f(a) = b$, then
  - b is the *image* of a under f;
  - a is the *preimage* of b.
- $f(\boldsymbol{A})$, the *range* of f, is the set of all images.
- Two functions are *equal* when they
  - have the same domain A and codomain B
  - map each element of A to the same element of B.

Representing Functions

Functions may be specified in different ways, e.g.,

- An explicit statement or diagram of the assignment.

  Students and grades example.

- A formula.

  $f(x) = x + 1$

- A computer program.

  A Java program that when given $n \in \mathbf{Z}$, produces the n^{th} Fibonacci Number (sect 2.4 and also Chapter 5).

Questions

$f(a) = ?$ z

The image of d is ? z

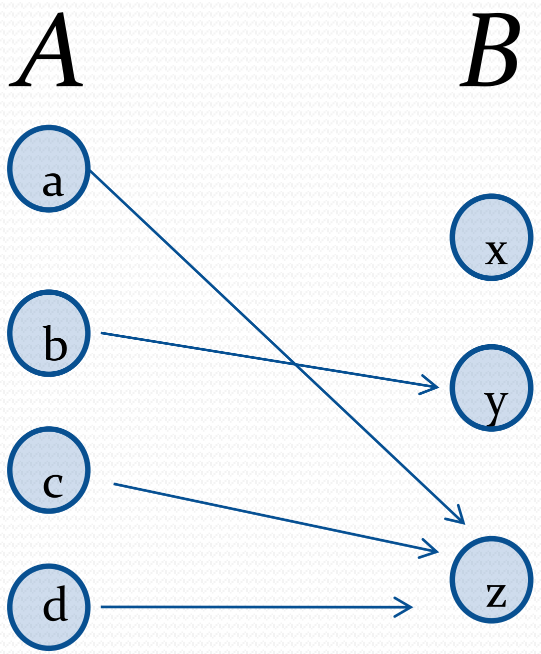
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A) = ?$ {y,z}

The preimage(s) of z is (are) ? {a,c,d}



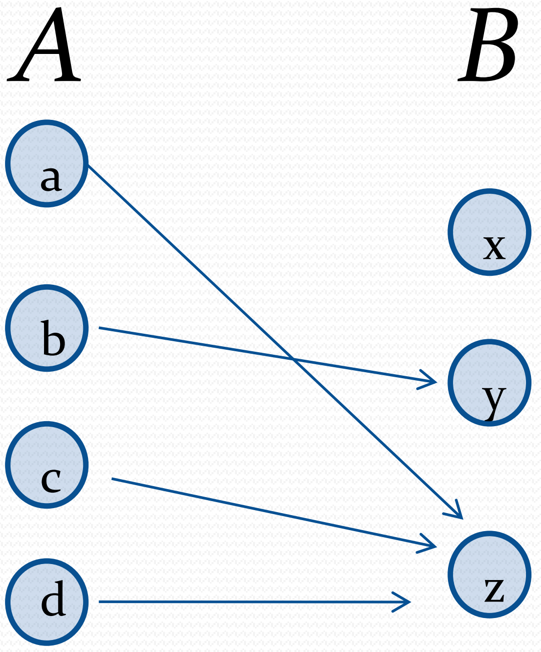
Question on Functions and Sets

- If $f : A \rightarrow B$ and S is a subset of A, then

$$f(S) = \{f(s) | s \in S\}$$

f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}

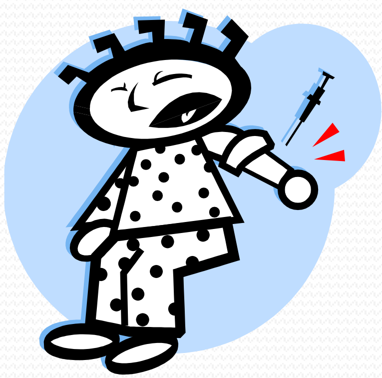


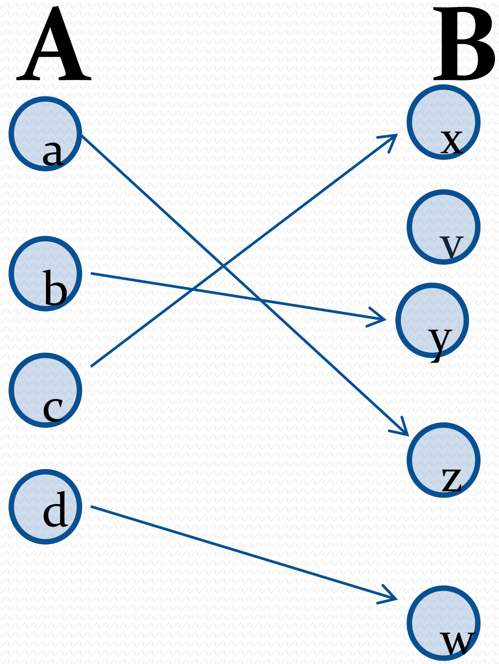
Injections

Definition: A function f is *one-to-one,* or *injective:*

$$\text{if } \forall a, b \in A, f(a) = f(b) \rightarrow a = b.$$

A function is an *injection* if it is one-to-one.





Surjections

Definition: A function f from A to B is *onto* or *surjective*, if $\forall\, b \in B, \exists\, a \in A$ with $f(a) = b$.

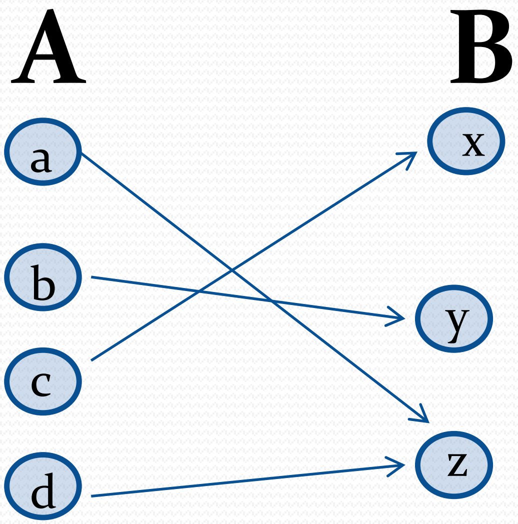
Or equivalently:

Is every $b \in B$ an image?

Do range and codomain coincide?

Does f(A) = B?

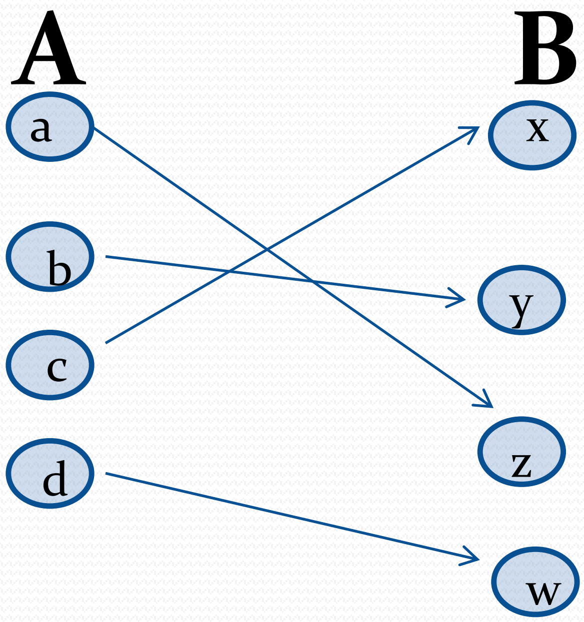
A function f is a *surjection* if it is *onto*.



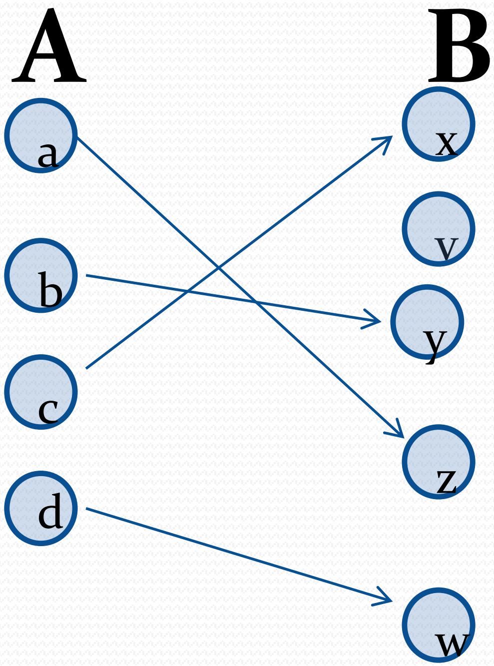
Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*,

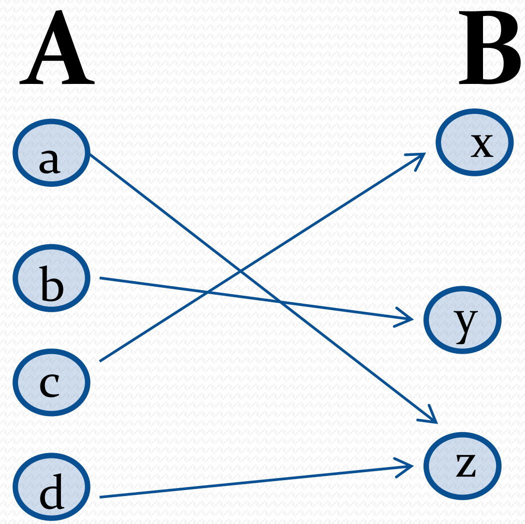
if it is both one-to-one and onto (surjective and injective).



Bijections: non-examples



Not onto. Why?



Not injective. Why?

Showing that *f* is one-to-one or onto

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Determining whether *f* is 1-1/onto

Let $f : \{a,b,c,d\} \rightarrow \{1,2,3\}$ defined by

$$f(a) = 3, f(b) = 2, f(c) = 1, f(d) = 3$$

Is *f* 1-1?

Solution: No, *f* is not 1-1 since $f(a) = f(d) = 3$.

In fact, just noting cardinalities of domain & codomain, we know that the function is not 1-1. (why?)

This is an instance of the **pigeonhole principle** that we encountered earlier.

Determining whether *f* is 1-1/onto

Let $f : \{a,b,c,d\} \rightarrow \{1,2,3\}$ defined by

$$f(a) = 3, f(b) = 2, f(c) = 1, f(d) = 3$$

Is *f* an onto function?

Solution: Yes, *f* is onto since all three elements of the codomain are images of elements in the domain.

If the codomain were changed to {1,2,3,4}, *f* would not be onto. (why?)

Determining whether f is 1-1/onto

Is $f : R \rightarrow R,\ f : x \mapsto x^2$ onto?

Solution: No, f is not onto because there is no real number x with $x^2 = -1$, for example.

How can we restrict the codomain so that f is onto?

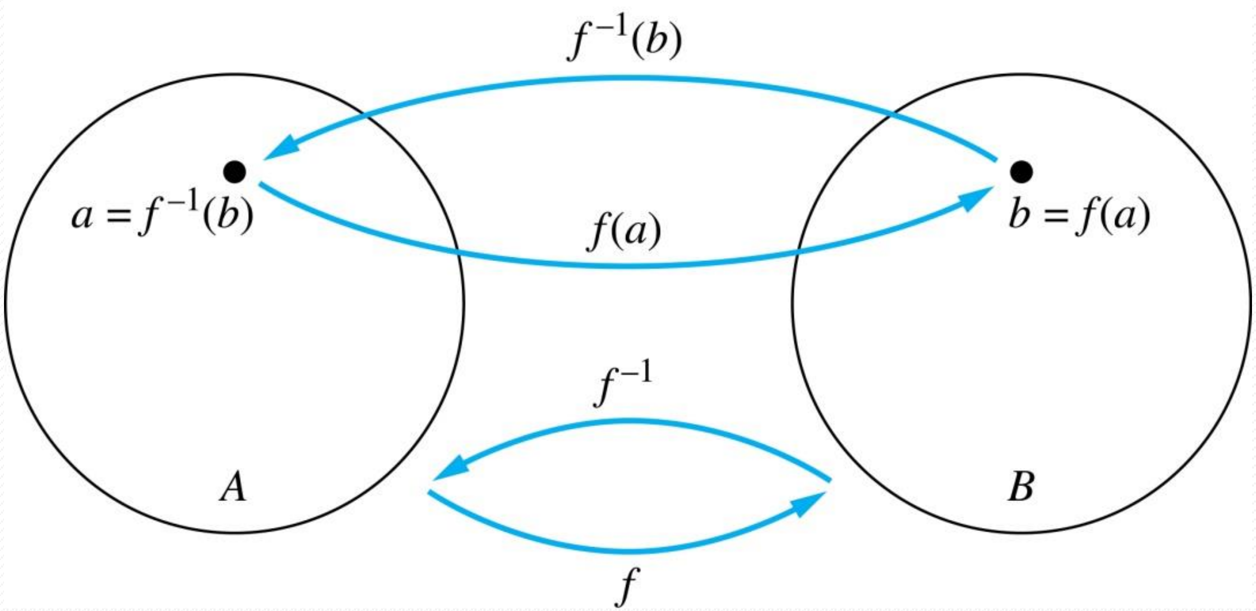
Solution: (works in general) Restrict the codomain to be the image of the domain, i.e.,

let codomain = $\mathbf{R}^{+} \cup \{0\}$.

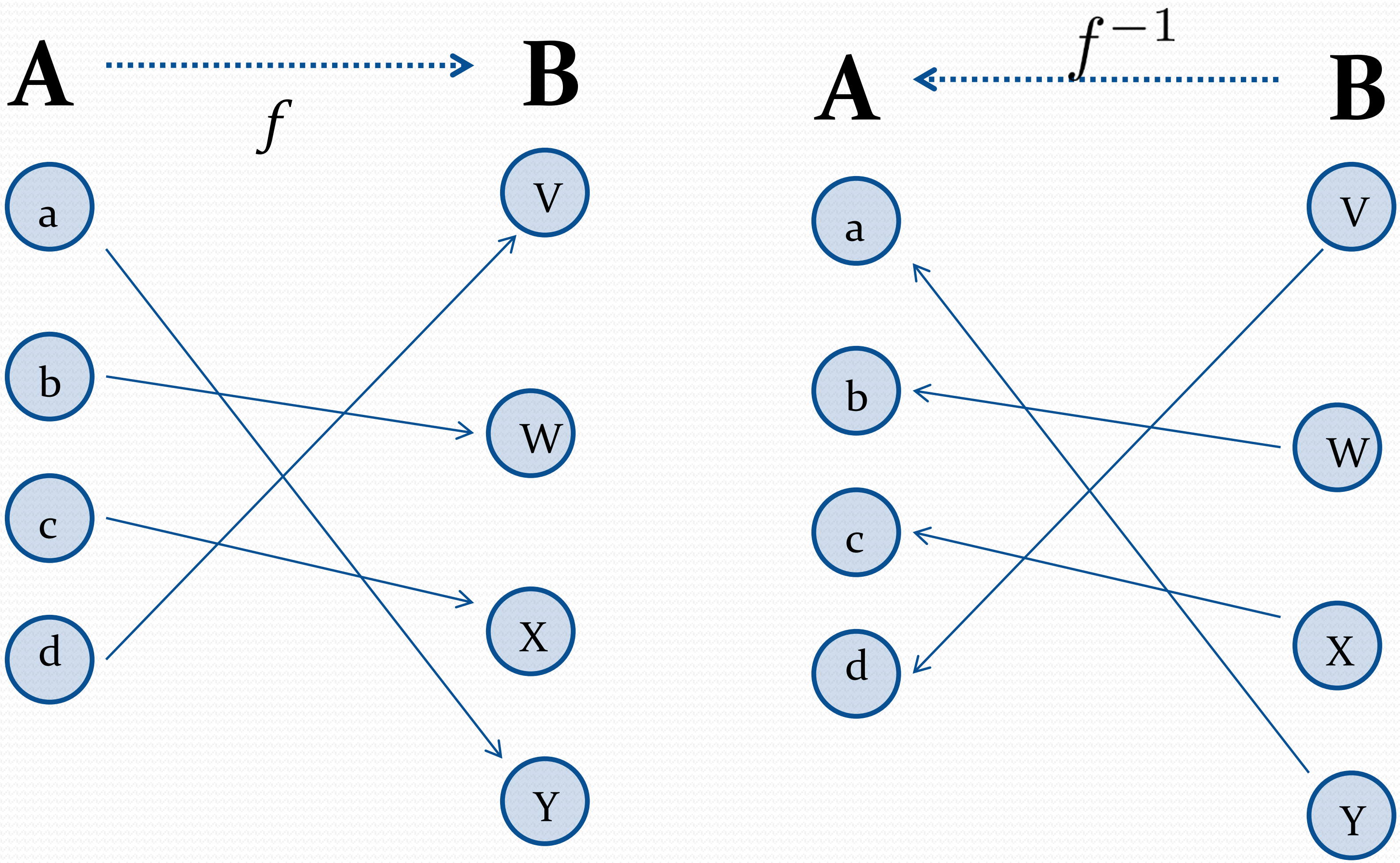
Inverse Functions

Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$

No inverse exists unless *f* is a bijection. **Exercise**: Why?



Inverse Functions



Questions

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: f is invertible because it is a bijection.
f^{-1} reverses the correspondence given by f, so

$$f^{-1}(1) = c, \quad f^{-1}(2) = a, \quad f^{-1}(3) = b.$$

Questions

Example 2: Let f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f : x \mapsto x + 1$.

Is f invertible, and if so, what is its inverse?

Solution: f is invertible because it is a bijection.

The inverse function f^{-1} reverses the correspondence so

$$f^{-1}(y) = y - 1.$$

Questions

Example 3: Let *f*: **R** → **R**, $f: x \mapsto x^2$ Is *f* invertible, and if so, what is its inverse?

Solution: *f* is not invertible because it is not 1-1.

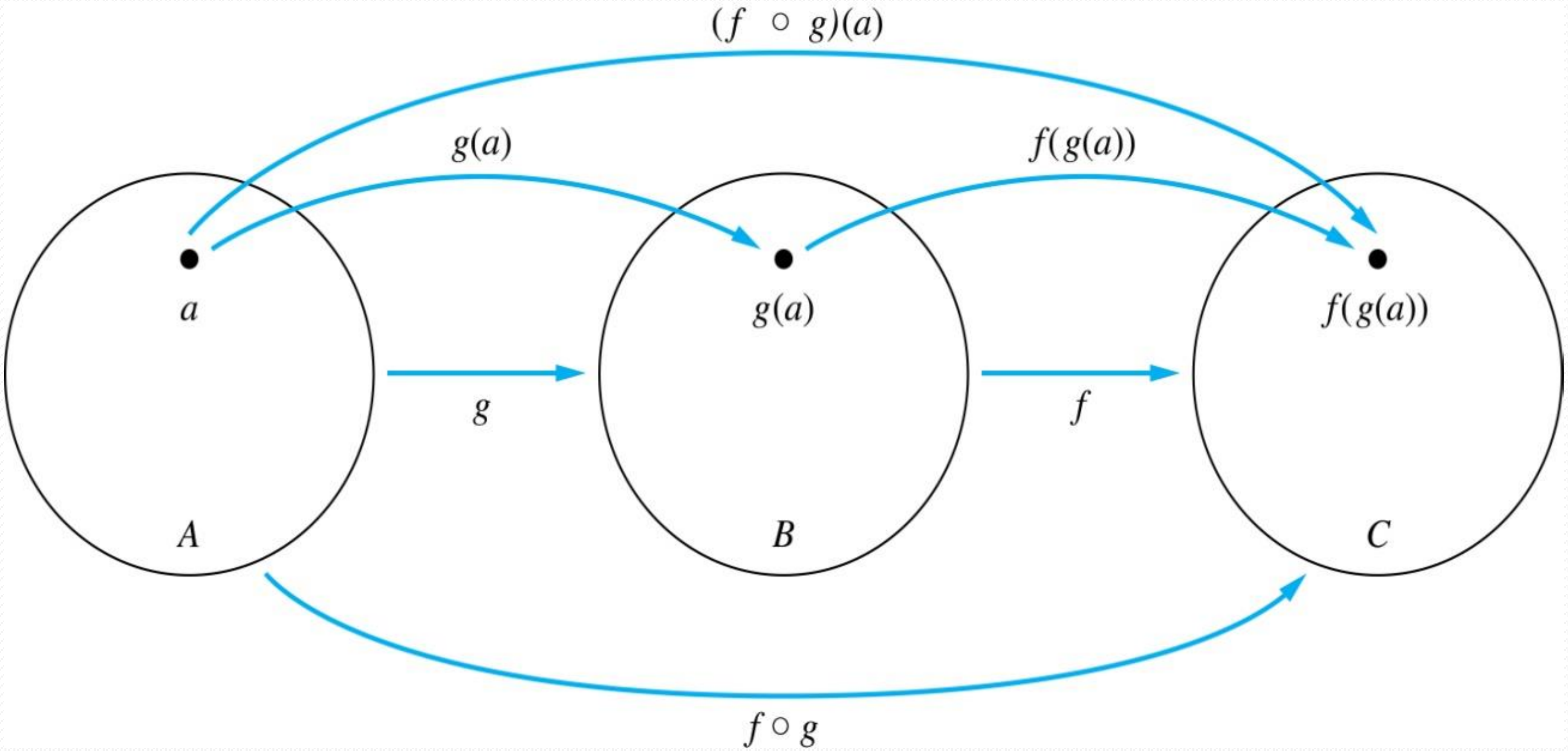
It is also not onto.

- The lack of ontoness can be solved by restricting the codomain to the functions image: $\mathbf{R}^+ \cup \{0\}$.
- To achieve 1-1 ness, we restrict the domain to
  - $\mathbf{R}^+ \cup \{0\}$ or $\mathbf{R}^- \cup \{0\}$,
- The resulting inverse functions are the positive and negative branches of the square root function, respectively.
- Visually, the 2 branches each satisfy the

  Vertical (for function) **and** horizontal (for 1-1) line tests.

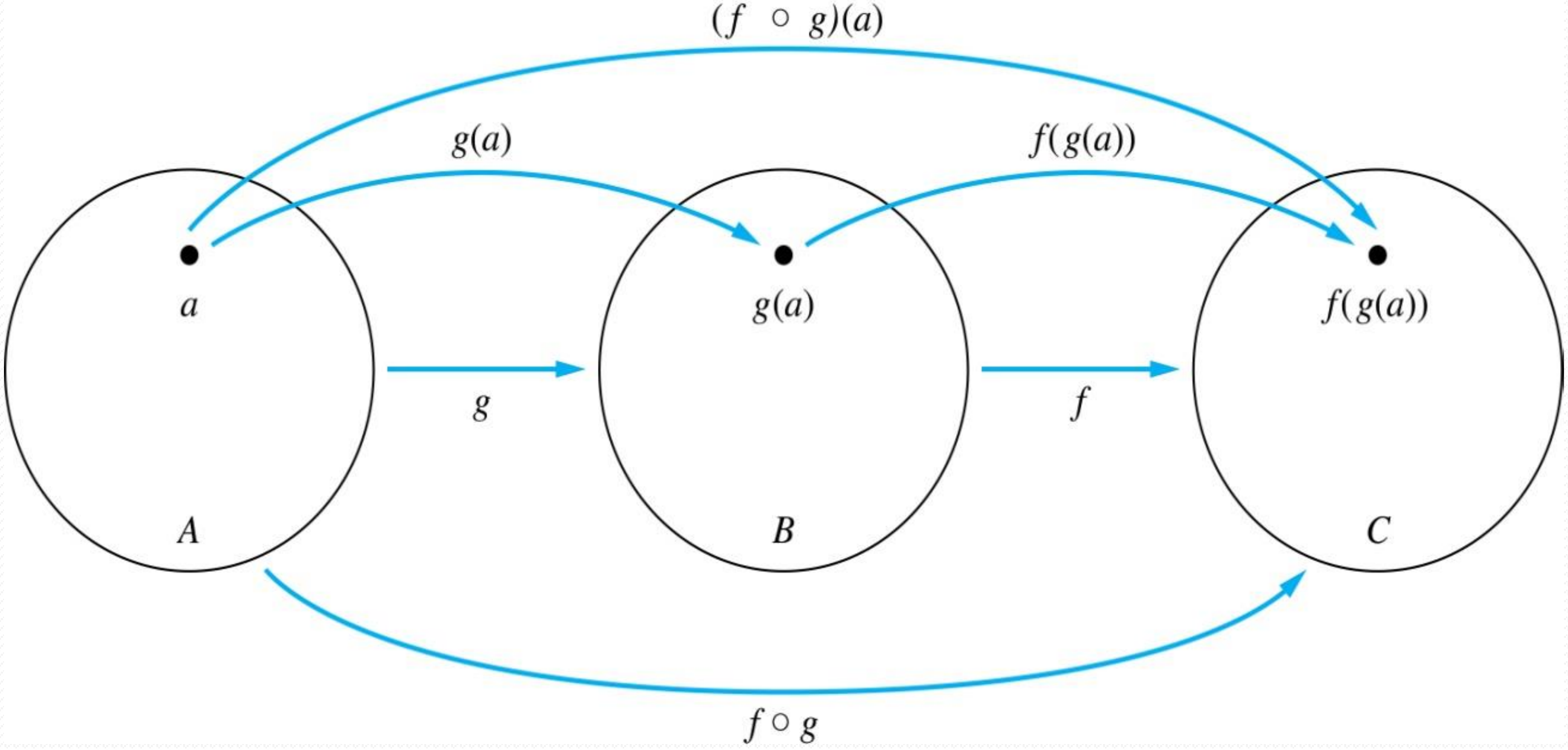
Composition

- **Definition**: Let f: $B \to C$, g: $A \to B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by $$f \circ g(x) = f(g(x))$$

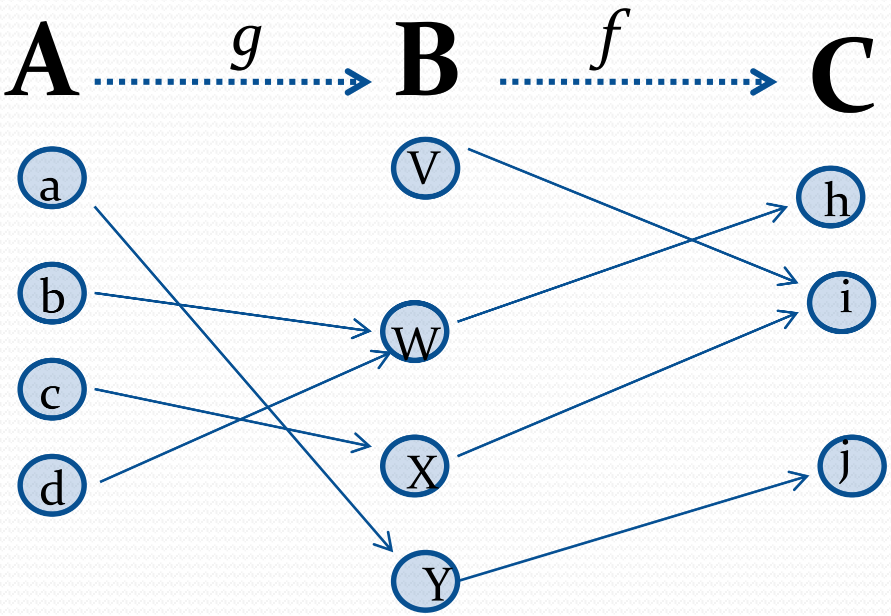


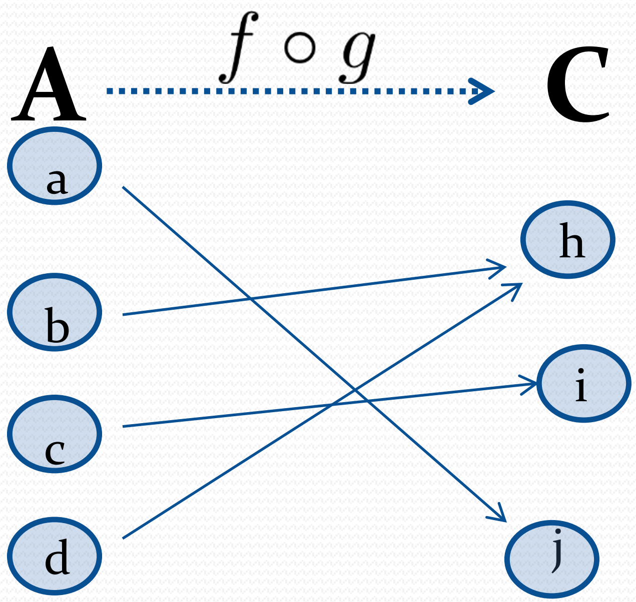
Composition

Note: range of g must be within the domain of f, i.e., $G(A) \subseteq B$, where B is the domain of f.



Composition





Composition

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$, then

$$f(g(x)) = (2x + 1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

We can conclude that composition in general is **not commutative**.

Composition Questions

Example 1: Let $g: \{a,b,c\} \rightarrow \{a,b,c\}$ such that

$$g(a) = b,\ g(b) = c, \text{ and } g(c) = a.$$

Let $f: \{a,b,c\} \rightarrow \{1,2,3\}$ such that

$$f(a) = 3,\ f(b) = 2, \text{ and } f(c) = 1.$$

What is $f \circ g$ and $g \circ f$?

Solution: The composition $f \circ g$ is defined by

$f \circ g\ (a) = f(g(a)) = f(b) = 2.$

$f \circ g\ (b) = f(g(b)) = f(c) = 1.$

$f \circ g\ (c) = f(g(c)) = f(a) = 3.$

$g \circ f$ not defined (range f is not contained in domain g).

Composition Questions

Example 2: Let $f, g: Z \to Z$ defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is $f \circ g$ and $g \circ f$?

Solution:

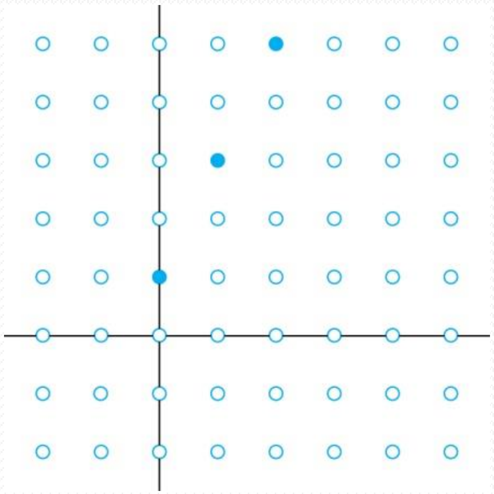
$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$g \circ f\ (x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

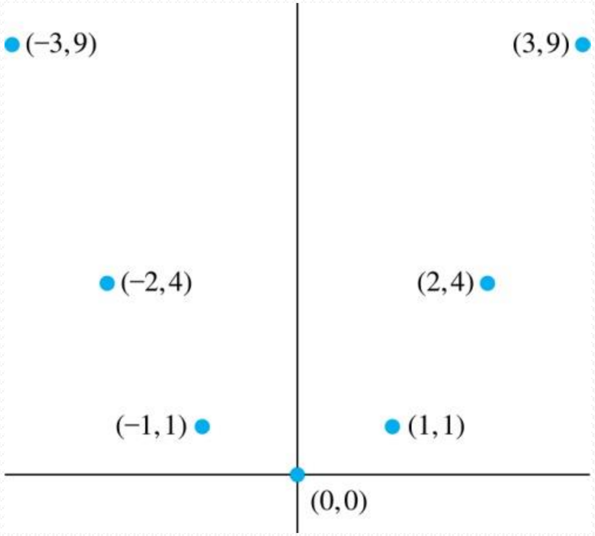
Again, note how composition is non-commutative in general.

Graphs of Functions

- Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$ from N to Z^+



Graph of $f(x) = x^2$ from Z to N

Some Important Functions

- The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

  is the largest integer less than or equal to *x*.

- The *ceiling* function, denoted

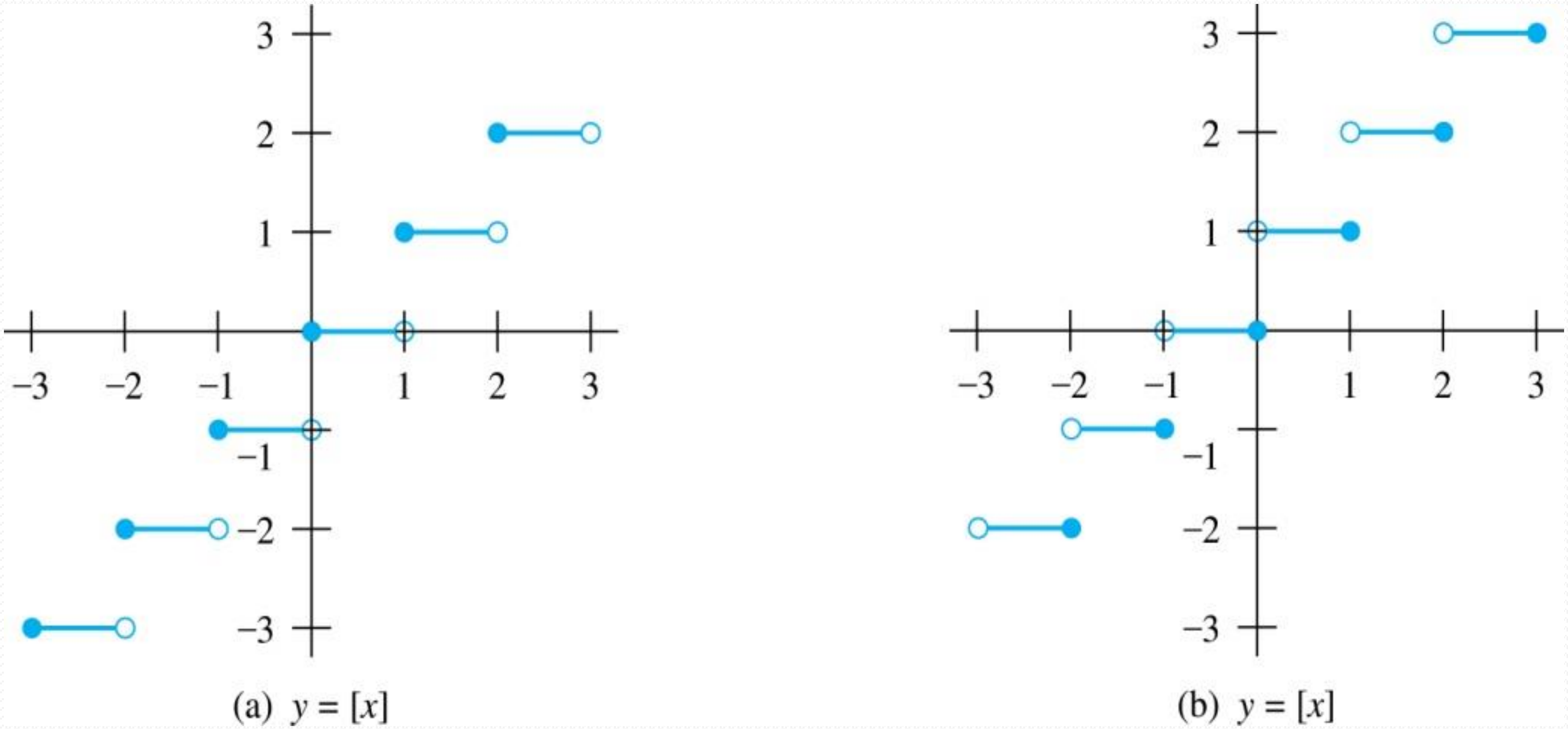
$$f(x) = \lceil x \rceil$$

  is the smallest integer greater than or equal to *x*.

Examples: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



Graph of (a) Floor and (b) Ceiling Functions

Floor and Ceiling Functions

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proving Properties of Functions

Example: Prove that x is a real number, then

$$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$$

Solution: Let $x = n + \varepsilon$, where n is an integer and $0 \le \varepsilon < 1$.

Case 1: $\varepsilon < ½$

- $2x = 2n + 2\varepsilon$ and $\lfloor 2x \rfloor = 2n$, since $0 \le 2\varepsilon < 1$.
- $\lfloor x + 1/2 \rfloor = n$, since $x + ½ = n + (1/2 + \varepsilon)$ and $0 \le ½ + \varepsilon < 1$.
- Hence, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n$.

Case 2: $\varepsilon \ge ½$

- $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$ and $\lfloor 2x \rfloor = 2n + 1$, since $0 \le 2\varepsilon - 1 < 1$.
- $\lfloor x + 1/2 \rfloor = \lfloor n + (1/2 + \varepsilon) \rfloor = \lfloor n + 1 + (\varepsilon - 1/2) \rfloor = n + 1$ since $0 \le \varepsilon - 1/2 < 1$.
- Hence, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1$. ◀

Factorial Function

Definition: f: $\mathbf{N} \rightarrow \mathbf{Z}^+$, denoted by $f(n) = n!$ is the product of the first n positive integers (or 1, if n = 0).

$f(n) = 1 \cdot 2 \cdots (n - 1) \cdot n$

$f(0) = 0! = 1$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\rightarrow\infty} f(n)/g(n) = 1$$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$