Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- Matrices
  - Matrix Arithmetic

Set Operations

Section 2.2

Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

Boolean Algebra

- Propositional calculus and set theory are both instances of *Boolean Algebra* (discussed in Chapter 12, but NOT formally part of either MAT 2440 or 2540).
- The operators in set theory are analogous to the corresponding operator in propositional calculus.
- As always there must be a universal set U. All sets are assumed to be subsets of U.

Union

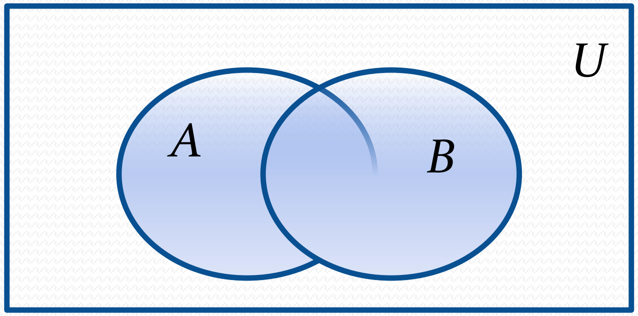
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

  Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



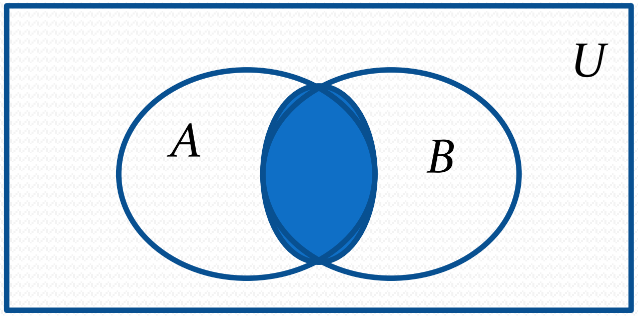
Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is
$$\{x | x \in A \wedge x \in B\}$$
- If intersection is empty, then *A* and *B* are *disjoint* (also *mutually exclusive*).
- **Example**: What is $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$
- **Example:** $\{1,2,3\} \cap \{4,5,6\} =?$

  Solution: ∅

Venn Diagram for $A \cap B$



Difference

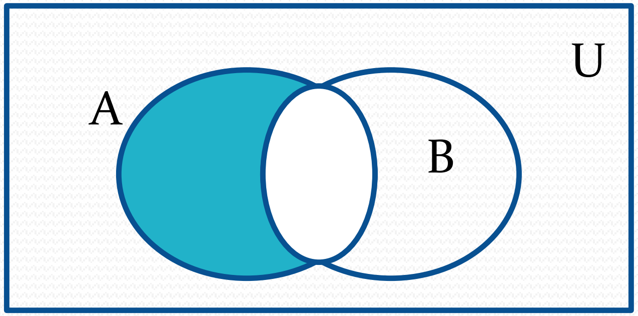
- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

- **Example**: What is $\{1, 2, 3\} - \{3, 4, 5\}$?

  Solution: $\{1, 2\}$

Venn Diagram for $A - B$



Complement

Definition: If A is a set, then the complement of A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

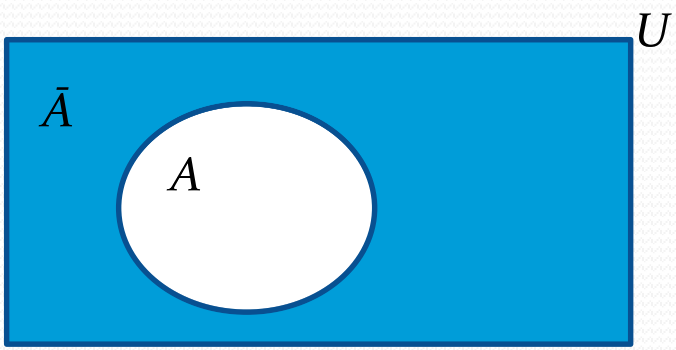
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U = R, find complement of $\{x \mid x > 70\}$.

Solution: $\{x \mid x \leq 70\}$

Venn Diagram for Complement



Symmetric Difference

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

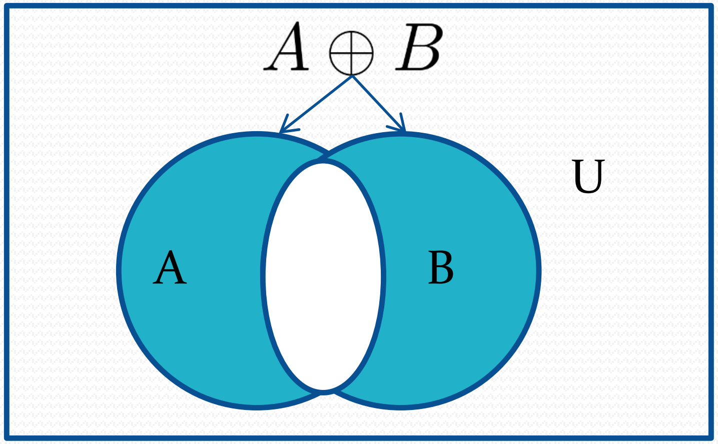
$$(A - B) \cup (B - A)$$

Example:

A = {1,2,3,4,5} B ={4,5,6,7,8}

What is $A \oplus B$?

- **Solution**: {1,2,3,6,7,8}



Venn Diagram

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

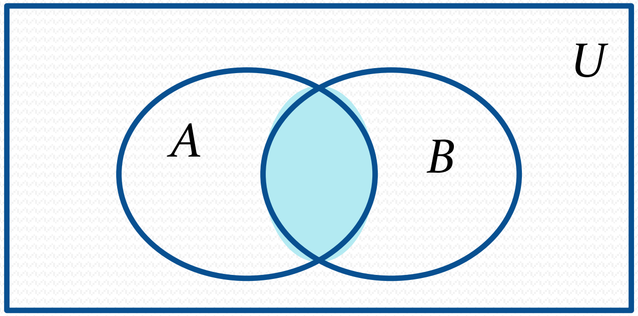
7. $A \oplus B$

   Solution: $\{1,2,3,6,7,8\}$

Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$



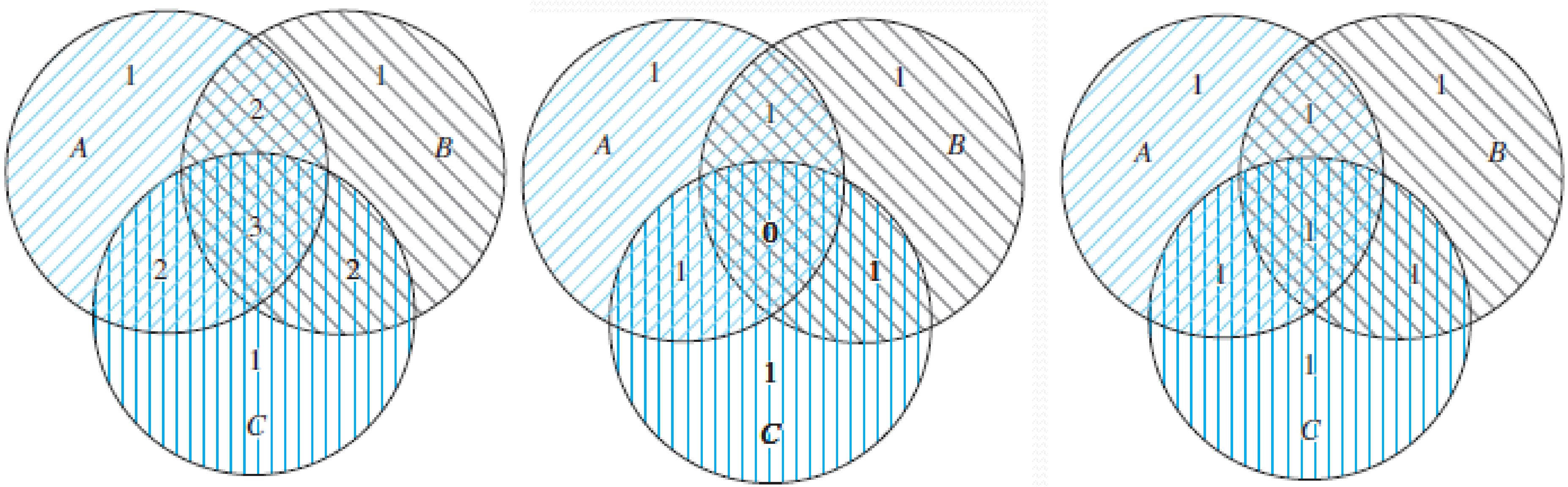
Venn Diagram for A, B, $A \cap B$, $A \cup B$

Example:

- Let A = math majors and B = CS majors.
- To count #students who are either math or CS majors:
  - add #math majors and #CS majors
  - subtract #joint CS/math majors.

Inclusion-Exclusion for 3 sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Set Identities (1 of 3)

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

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Set Identities (2 of 3)

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Set Identities (3 of 3)

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption law

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

Ways to Prove Set Identities

1. Show each set (side of the identity) is subset of other.
2. Use set builder notation and propositional logic.
3. Membership Tables:
   - Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity.
   - Use 1 to indicate it is in the set and a 0 to indicate that it is not.

2^{nd} De Morgan Law: containment

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

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2^{nd} De Morgan Law: containment

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

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2nd De Morgan Law: containment

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

2^{nd} De Morgan Law: Set-Builder Notation

$$
\begin{aligned}
&\overline{A \cap B} \\
= \quad & \{x | x \notin A \cap B\} && \text{by defn. of complement} \\
= \quad & \{x | \neg(x \in (A \cap B))\} && \text{by defn. of does not belong symbol} \\
= \quad & \{x | \neg(x \in A \wedge x \in B\} && \text{by defn. of intersection} \\
= \quad & \{x | \neg(x \in A) \vee \neg(x \in B)\} && \text{by 1st De Morgan law} \\
& && \text{for Prop Logic} \\
= \quad & \{x | x \notin A \vee x \notin B\} && \text{by defn. of not belong symbol} \\
= \quad & \{x | x \in \overline{A} \vee x \in \overline{B}\} && \text{by defn. of complement} \\
= \quad & \{x | x \in \overline{A} \cup \overline{B}\} && \text{by defn. of union} \\
= \quad & \overline{A} \cup \overline{B} && \text{by meaning of notation}
\end{aligned}
$$

distributive law: Membership Table

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions

Let A_1, A_2 ,…, A_n be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

This is well-defined, since union is associative.

- For i = 1,2,…, let A_i = {i, i + 1, i + 2, ….}. Then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

Generalized Intersections

Let A_1, A_2 ,...., A_n be an indexed collection of sets.

We define:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

This is well-defined, since intersection is associative.

- For i = 1,2,..., let A_i = {i, i + 1, i + 2, ...}. Then,

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots..\} = A_n$$