

# Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

# Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- Matrices
  - Matrix Arithmetic

# Sets

Section 2.1

# Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

# Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

# Sets

- A *set* is an **unordered** collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- $a \in A$  denotes that  $a$  is an element of the set  $A$ .
- If  $a$  is not a member of  $A$ , write  $a \notin A$

# Repeated elements, use of ellipses

- $S = \{a, b, c, d\}$

- Order not important

$$S = \{a, b, c, d\} = \{b, c, a, d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (...) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a, b, c, d, \dots, z\}$$

# Examples

- Set of all vowels in the English alphabet:

$$V = \{a, e, i, o, u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1, 3, 5, 7, 9\}$$

- Set of all positive integers less than 100:

$$S = \{1, 2, 3, \dots, 99\}$$

- Set of all integers less than 0:

$$S = \{\dots, -3, -2, -1\}$$



# Some Important Sets

$\mathbf{N}$  = *natural numbers* =  $\{0,1,2,3,\dots\}$

$\mathbf{Z}$  = *integers* =  $\{\dots,-3,-2,-1,0,1,2,3,\dots\}$

$\mathbf{Z}^+$  = *positive integers* =  $\{1,2,3,\dots\}$

$\mathbf{R}$  = *set of real numbers*

$\mathbf{R}^+$  = *set of positive real numbers*

$\mathbf{C}$  = *set of complex numbers.*

$\mathbf{Q}$  = *set of rational numbers*

$\mathbf{R} \setminus \mathbf{Q}$  = *set of irrational numbers*

# Set-Builder Notation

- Specify the property or properties that all members must satisfy:

$$S = \{x \mid x \in \mathbf{Z}^+ \text{ and } x < 100\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$$

- A predicate may be used:

$$S = \{x \mid P(x)\}$$

- Example:  $S = \{x \mid \text{Prime}(x)\}$

- Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$$

# Interval Notation

$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

# Interval Notation

$[a,b] = \{x \mid a \leq x \leq b\}$  closed!

$[a,b) = \{x \mid a \leq x < b\}$  half-open (or half-closed!)

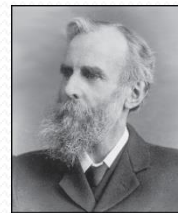
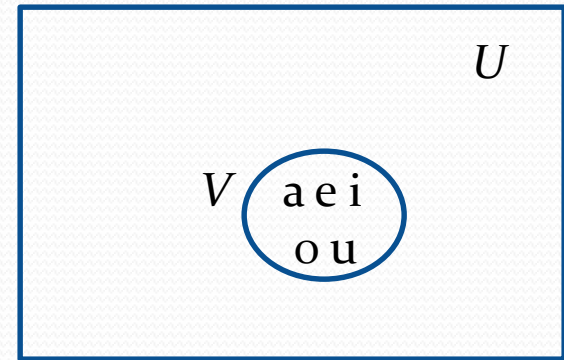
$(a,b] = \{x \mid a < x \leq b\}$  ditto

$(a,b) = \{x \mid a < x < b\}$  open!

# Universal Set and Empty Set

- The *universal set*  $U$  is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements.  
(Symbolized  $\emptyset$ , but  $\{\}$  also used.)

Venn Diagram



John Venn (1834-1923)  
Cambridge, UK

# Russell's Paradox

- Let  $S$  be the set of all sets which are not members of themselves.
- A paradox results from trying to answer the question “Is  $S$  a member of itself?”
- Related Paradox:
  - Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question “Does Henry shave himself?”



Bertrand Russell (1872-1970)  
Cambridge, UK  
Nobel Prize Winner

# Some things to remember

- Sets can be elements of sets.

$$\{\{1,2,3\}, a, \{b,c\}\}$$

$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$$

- The empty set is different from a set containing the empty set.

$$\emptyset \neq \{ \emptyset \}$$

# Set Equality

**Definition:** Two sets are *equal* if and only if they have the same elements.

- If  $A$  and  $B$  are sets,

then  $A$  and  $B$  are equal if and only if

• 
$$\forall x(x \in A \leftrightarrow x \in B)$$

- We write  $A = B$  if  $A$  and  $B$  are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$



# Subsets

**Definition:** The set  $A$  is a *subset* of  $B$ , if and only if every element of  $A$  is also an element of  $B$ .

- The notation  $A \subseteq B$  is used to indicate that  $A$  is a subset of the set  $B$ .
- $A \subseteq B$  holds if and only if  $\forall x(x \in A \rightarrow x \in B)$  is true.
  1. Because  $a \in \emptyset$  is always false,  $\emptyset \subseteq S$ , for every set  $S$ .
  2. Because  $a \in S \rightarrow a \in S$ ,  $S \subseteq S$ , for every set  $S$ .

# Showing a Set is or is not a Subset

- **Showing that A is a Subset of B:** To show that  $A \subseteq B$ , show that if  $x$  belongs to  $A$ , then  $x$  also belongs to  $B$ .
- **Showing that A is not a Subset of B:** To show that  $A$  is not a subset of  $B$ ,  $A \not\subseteq B$ , find an element  $x \in A$  with  $x \notin B$ . (Such an  $x$  is a counterexample to the claim that  $x \in A$  implies  $x \in B$ .)

## Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

# Another look at Equality of Sets

- Recall that two sets  $A$  and  $B$  are *equal*, denoted by  $A = B$ , iff 
$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that  $A = B$  iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

# Proper Subsets

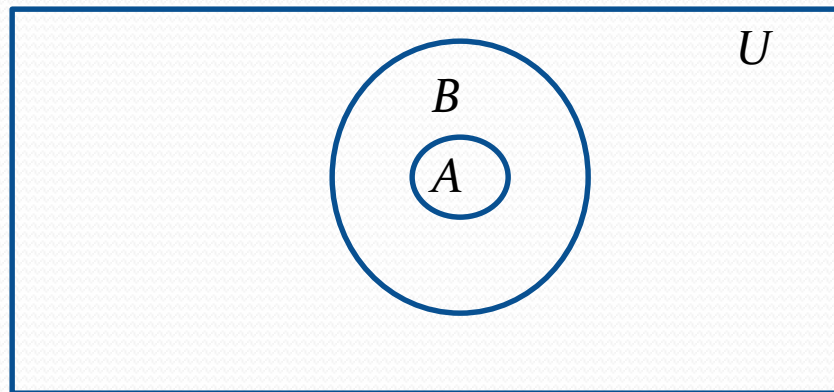
**Definition:** If  $A \subseteq B$ , but  $A \neq B$ , then we say  $A$  is a *proper subset* of  $B$ , denoted by  $A \subset B$ .

If  $A \subset B$ , then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



# Set Cardinality

**Definition:** If  $\exists n \in \mathbf{N}$  distinct elements in  $S$ , we say that  $S$  is *finite*. Otherwise it is *infinite*. The *cardinality* of set  $A$  is denoted by  $|A|$ . If  $A$  is a finite, then  $|A|$  is this same  $n$ .

**Examples:**

1.  $|\emptyset| = 0$
2. Let  $S$  be the letters of the English alphabet. Then  $|S| = 26$
3.  $|\{1,2,3\}| = 3$
4.  $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

# Power Sets

**Definition:** The set of all subsets of a set  $A$ , denoted  $\mathcal{P}(A)$ , is called the *power set* of  $A$ .

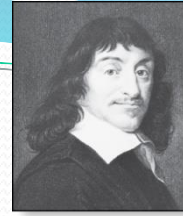
**Example:** If  $A = \{a, b\}$  then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

- If a set has  $n$  elements, then the cardinality of the power set is  $2^n$ . (In Chapters 5 and 6, we will discuss different ways to show this.)

# Tuples

- The *ordered n-tuple*  $(a_1, a_2, \dots, a_n)$  is the ordered collection that has  $a_1$  as its first element and  $a_2$  as its second element and so on until  $a_n$  as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs  $(a, b)$  and  $(c, d)$  are equal if and only if  $a = c$  and  $b = d$ .



René Descartes  
(1596-1650)

# Cartesian Product

**Definition:** The *Cartesian Product* of two sets  $A$  and  $B$ , denoted by  $A \times B$  is the set of ordered pairs  $(a,b)$  where  $a \in A$  and  $b \in B$ .

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

**Example:**

$$A = \{a, b\} \quad B = \{1, 2, 3\}$$

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$

- **Definition:** A subset  $R$  of the Cartesian product  $A \times B$  is called a *relation* from the set  $A$  to the set  $B$ . (Relations will be covered in depth in Chapter 9. )



# Cartesian Product

**Definition:** The cartesian products of the sets  $A_1, A_2, \dots, A_n$ , denoted by  $A_1 \times A_2 \times \dots \times A_n$ , is the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  where  $a_i$  belongs to  $A_i$  for  $i = 1, \dots, n$ .

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

**Example:** What is  $A \times B \times C$  where  $A = \{0,1\}$ ,  $B = \{1,2\}$  and  $C = \{0,1,2\}$

**Solution:**  $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\}$

# Truth Sets of Quantifiers

- Given a predicate  $P$  and a domain  $D$ , we define the *truth set* of  $P$  to be the set of elements in  $D$  for which  $P(x)$  is true. The truth set of  $P(x)$  is denoted by

$$\{x \in D \mid P(x)\}$$

- **Example:** The truth set of  $P(x)$  where the domain is the integers and  $P(x)$  is “ $|x| = 1$ ” is the set  $\{-1, 1\}$