

The Foundations: Logic and Proofs

Chapter 1, Part III: Proofs

With Question/Answer Animations

Summary

- Valid Arguments and Rules of Inference
- Proof Methods
- Proof Strategies

Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
- Using Rules of Inference to Build Arguments
- Rules of Inference for Quantified Statements
- Building Arguments for Quantified Statements

Revisiting the Socrates Example

- We have the two premises:
 - “All men are mortal.”
 - “Socrates is a man.”
- And the conclusion:
 - “Socrates is mortal.”
- How do we get the conclusion from the premises?

The Argument

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\forall x (Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore Mortal(Socrates)$$

- We will see shortly that this is a valid argument.

Valid Arguments

- The rules of inference are the essential building block in the construction of valid arguments.
 1. Propositional Logic
Inference Rules
 2. Predicate Logic
Inference rules for propositional logic plus additional inference rules for variables & quantifiers.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions.
 - All but the final proposition are called *premises*.
 - The last statement is the *conclusion*.
- The argument is valid if the premises imply the conclusion.
- If the premises are p_1, p_2, \dots, p_n & the conclusion is q then $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.
- Inference rules are simple argument forms that will be used to construct more complex argument forms.

Rules of Inference: Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

Example:

Let p be “It is snowing.”

Let q be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”

“It is snowing.”

“Therefore, I will study discrete math.”

Modus Tollens (contrapositive)

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Corresponding Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

Example:

Let p be “it is snowing.”

Let q be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”

“I will not study discrete math.”

“Therefore, it is not snowing.”

Hypothetical Syllogism (transitive)

$$p \rightarrow q$$

$$q \rightarrow r$$

$$\therefore p \rightarrow r$$

Corresponding Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let p be “it snows.”

Let q be “I will study discrete math.”

Let r be “I will get an A.”

“If it snows, then I will study discrete math.”

“If I study discrete math, I will get an A.”

“Therefore, If it snows, I will get an A.”

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Corresponding Tautology:
 $(\neg p \wedge (p \vee q)) \rightarrow q$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math or I will study English literature.”

“I will not study discrete math.”

“Therefore , I will study English literature.”

Addition

$$\frac{p}{\therefore p \vee q}$$

Corresponding Tautology:
 $p \rightarrow (p \vee q)$

Example:

Let p be “I will study discrete math.”

Let q be “I will visit Las Vegas.”

“I will study discrete math.”

“Therefore, I will study discrete math or I will visit Las Vegas.”

Simplification

$$\frac{p \wedge q}{\therefore q}$$

Corresponding Tautology:
 $(p \wedge q) \rightarrow p$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math and English literature”

“Therefore, I will study discrete math.”

Conjunction

$$\frac{p}{q} \\ \hline \therefore p \wedge q$$

Corresponding Tautology:
 $((p) \wedge (q)) \rightarrow (p \wedge q)$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math.”

“I will study English literature.”

“Therefore, I will study discrete math and I will study English literature.”

Resolution

Resolution plays an important role in AI and is used in Prolog.

$$\frac{\neg p \vee r \quad p \vee q}{\therefore q \vee r}$$

Corresponding Tautology:
 $((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$

Example:

Let p be “I will study discrete math.”

Let r be “I will study English literature.”

Let q be “I will study databases.”

“I will not study discrete math or I will study English literature.”

“I will study discrete math or I will study databases.”

“Therefore, I will study databases or I will study English literature.”

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements.
- Each statement is either a premise or follows from previous statements by rules of inference.
- The last statement is called conclusion.

S_1

S_2

·

·

·

S_n

$\therefore C$

Valid Arguments

Example 1: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step	Reason
1. $p \wedge (p \rightarrow q)$	Premise
2. p	Simplification using (1)
3. $p \rightarrow q$	Simplification using (1)
4. q	Modus Ponens using (2) and (3)

Valid Arguments

Example 2:

- With these hypotheses:
 - “It is not sunny this afternoon and it is colder than yesterday.”
 - “We will go swimming only if it is sunny.”
 - “If we do not go swimming, then we will take a canoe trip.”
 - “If we take a canoe trip, then we will be home by sunset.”
- Using the inference rules, construct a valid argument for the conclusion:
 - “We will be home by sunset.”

Solution:

1. Choose propositional variables:
 - p : “It is sunny this afternoon.” r : “We will go swimming.” t : “We will be home by sunset.”
 - q : “It is colder than yesterday.” s : “We will take a canoe trip.”
2. Translation into propositional logic:

Hypotheses: $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s, s \rightarrow t$

Conclusion: t

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Valid Arguments

3. Construct the Valid Argument

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- Each statement is either a premise or follows from previous statements by Rules of Inference for:
 - Propositional Logic
 - Quantified Statements

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cuddly.”

“Therefore, Fido is cuddly.”

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

“There is someone who got an A in the course.”

“Let’s call her a and say that a got an A”

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

“Michelle got an A in the class.”

“Therefore, someone got an A in the class.”

Using Rules of Inference

Example 1: Using the rules of inference, construct a valid argument to show that

“John Smith has two legs”

is a consequence of the premises:

“Every man has two legs.” “John Smith is a man.”

Solution: Let $M(x)$ denote “ x is a man” and $L(x)$ “ x has two legs” and let John Smith be a member of the domain.

Valid Argument:

Step	Reason
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

Example 2: Use the rules of inference to construct a valid argument showing that the conclusion

“Someone who passed the first exam has not read the book.”

follows from the premises

“A student in this class has not read the book.”

“Everyone in this class passed the first exam.”

Solution: Let $C(x)$ denote “ x is in this class,” $B(x)$ denote “ x has read the book,” and $P(x)$ denote “ x passed the first exam.”

First we translate the premises and conclusion into symbolic form.

$$\frac{\begin{array}{l} \exists x(C(x) \wedge \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \end{array}}{\therefore \exists x(P(x) \wedge \neg B(x))}$$

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Using Rules of Inference

Valid Argument:

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\forall x(Man(x) \rightarrow Mortal(x))$$
$$Man(Socrates)$$

$$\therefore Mortal(Socrates)$$

Solution for Socrates Example

Valid Argument

Step

1. $\forall x(Man(x) \rightarrow Mortal(x))$

2. $Man(Socrates) \rightarrow Mortal(Socrates)$

3. $Man(Socrates)$

4. $Mortal(Socrates)$

Reason

Premise

UI from (1)

Premise

MP from (2)
and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\forall x(P(x) \rightarrow Q(x))$$

$P(a)$, where a is a particular
element in the domain

$$\therefore Q(a)$$

This rule could be used in the Socrates example.