MAT 2630 Halleck Fall 2015 Practice Exam 2 Solutions v2

REMINDER: your 2 page (front and back) 1 sheet hand-written set of formulas and notes will be 10% of your grade. Please do as much of the exam as you can by hand. However, you may use a calculator if you need it. The actual exam will consist of questions similar to 5 of the ones that you see below. Each question will be worth 18%.

1. Solve the system Ax=[2;4;6] by finding the LU factorization for the matrix A below and using the two-step back substitution. The factorization uses the naïve Gaussian elimination from 2.1. Work with the present pivot and use it to eliminate all the entries in the same column below it, then move to the next diagonal entry. Row permutations are not allowed. During the back substitutions, you can use the elementary row operation of scaling as well.

$$\begin{bmatrix} 2 & 2 & 3 \\ 4 & 2 & 0 \\ 4 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 3 \\ \boxed{2} & -2 & -6 \\ \boxed{2} & 0 & -4 \end{bmatrix} \to LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 0 & -2 & -6 \\ 0 & 0 & -4 \end{bmatrix}$$

First back substitution:

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 2 & 1 & 0 & | & 4 \\ 2 & 0 & 1 & | & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Second back substitution:

$$\begin{bmatrix} 2 & 2 & 3 & | & 2 \\ 0 & -2 & -6 & | & 0 \\ 0 & 0 & -4 & | & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & | & 7/2 \\ 0 & -2 & 0 & | & -3 \\ 0 & 0 & 1 & | & -1/2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & | & 1/2 \\ 0 & 1 & 0 & | & 3/2 \\ 0 & 0 & 1 & | & -1/2 \end{bmatrix}$$

So the solution is [1/4;3/2;-1/2]

2. For the system of equations: $x_1 - 2x_2 = 3$, $3x_1 - 4x_2 = 7$
   a. Find the condition number for the coefficient matrix. cond(A) = ||A|| ||A^{-1}||.
   $\begin{vmatrix} 1 & -2 \\ 3 & -4 \end{vmatrix} = -4 + 6 = 2$ so $A^{-1} = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix} / 2$ Hence ||A||=7 and ||A^{-1}||=3 so cond(A)=21
   Basically the conditioning number is high because the row vectors are close to each other in direction.

   b. Solve the system exactly.
   $$\begin{bmatrix} 1 & -2 & | & 3 \\ 3 & -4 & | & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 3 \\ 0 & 2 & | & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & -1 \end{bmatrix}$$

   c. Find the forward and backward errors and error magnification factor for the approximate solution [−2, −3].
   Forward error is ∞-norm of [1 -1]-[-2 -3]=[3 2] which is 3 and the relative forward error is also 3.
   r = b − Ax_a=[3 7]-[4 6]=[-1 1] so backward error is 1 and relative backwards error is 1/7
   The error magnification number is the ratio of the relative forward error to the relative backward error is 3/(1/7) which is 21.

Since the error magnification number is the same as the condition number, this means that our particular b is an example of that which maximizes the error magnification number. In another words, no other b will have a higher error magnification number with this A.

3. Find the PA= LU factorization for the matrix A below and check by matrix multiplication.
   Here we use the algorithm presented in section 2.4. The step of comparing all the potential pivots for a particular column is called partial pivoting.

$$\begin{bmatrix} e_1 & | & 1 & 2 & -3 \\ e_2 & | & 2 & 4 & 2 \\ e_3 & | & -1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} e_2 & | & 2 & 4 & 2 \\ e_1 & | & 1 & 2 & -3 \\ e_3 & | & -1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} e_2 & | & 2 & 4 & 2 \\ e_1 & | & \boxed{1/2} & 0 & -4 \\ e_3 & | & \boxed{-1/2} & 2 & 4 \end{bmatrix} \to \begin{bmatrix} e_2 & | & 2 & 4 & 2 \\ e_3 & | & \boxed{-1/2} & 2 & 4 \\ e_1 & | & \boxed{1/2} & 0 & -4 \end{bmatrix} \text{ so}$$

P=[0 1 0;0 0 1;1 0 0]

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 2 \\ -1 & 0 & 3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{bmatrix} \text{ Yes, they both are } \begin{bmatrix} 2 & 4 & 2 \\ -1 & 0 & 3 \\ 1 & 2 & -3 \end{bmatrix}$$

4. Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of the Gauss–Seidel Method from starting vector [0;0;0].

$$\begin{aligned} u - 8v - 2w &= 1 \\ u + v + 5w &= 4 \\ 3u - v + w &= -2 \end{aligned} \qquad \begin{aligned} 3u - v + w &= -2 \\ u - 8v - 2w &= 1 \\ u + v + 5w &= 4 \end{aligned} \qquad \begin{aligned} u &= (v - w - 2)/3 \\ v &= (u - 2w - 1)/8 \\ w &= (-u - v + 4)/5 \end{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} -2/3 \\ (-2/3-1)/8 = -5/24 \\ (2/3+5/24+4)/5 = 117/120 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} -2/3 \\ -5/24 \\ 117/120 \end{bmatrix} \to \begin{bmatrix} (-5/24 - 117/120 - 2)/3 = -382/360 = -191/180 \\ (-191/180 - 117/60 - 1)/8 = -722/(180*8) = -361/720 \\ (191/180 + 361/720 + 4)/5 = 801/720 = 89/80 \end{bmatrix}$$

5. Verify that the symmetric matrix A below is positive definite. Find the Cholesky factorization A = $R^T R$:

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$ One relatively easy way to show that matrix is positive definite is show that all of its leading principal minors are positive. The kth leading principal minor of a matrix M is the determinant of its upper-left k by k sub-matrix. This condition is known as Sylvester's criterion. For this problem the principal minors are 1, $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ and the matrix A itself. The determinants are 1, 2-1=1 and using sum of down diagonals minus sum of up diagonals, (4+1+1)-(2+1+2)=1, each of which is positive, so matrix is PD. For the Cholesky factorization, initialize R=A and i=1. In this iteration,

1. The entries below the diagonal in the first column become 0.
2. The first diagonal entry gets the square root applied to it. New diagonal entry for R is 1=√1.
3. The entries in first row beyond the diagonal are divided by this square root.

Let C be the matrix gotten from R by selecting the square matrix from the i+1st diagonal entry on so $C=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ and u=[-1 -1]/1=[-1 -1], and $u^T u=\begin{bmatrix}-1\\ -1\end{bmatrix}\begin{bmatrix}-1 & -1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$. Replace C with C-$u^T$u=
$\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}-\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Hence, after one iteration, $R=\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ which is already upper triangular so we stop.

As a check $R^T R=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & -1\\ -1 & 1+1 & 1\\ -1 & 1 & 1+1\end{bmatrix}=\begin{bmatrix}1 & -1 & -1\\ -1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix}=A$

6. Find the best line through (0,0), (1,3), (2,3), (5,6), and find the RMSE. Graph the points as well as the solution. Verify that $\sum_{i=1}^{4}(c_1+c_2t_i-y_i)^2/\sqrt{4}=RMSE$. System is $[A|b]=\begin{bmatrix}1 & 0 & | & 0\\ 1 & 1 & | & 3\\ 1 & 2 & | & 3\\ 1 & 5 & | & 6\end{bmatrix}$. Normalize the system by multiplying by A^T: $A^T[A|b]=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 5\end{bmatrix}\begin{bmatrix}1 & 0 & | & 0\\ 1 & 1 & | & 3\\ 1 & 2 & | & 3\\ 1 & 5 & | & 6\end{bmatrix}=\begin{bmatrix}4 & 8 & | & 12\\ 8 & 30 & | & 39\end{bmatrix}$. Solve the resulting system:

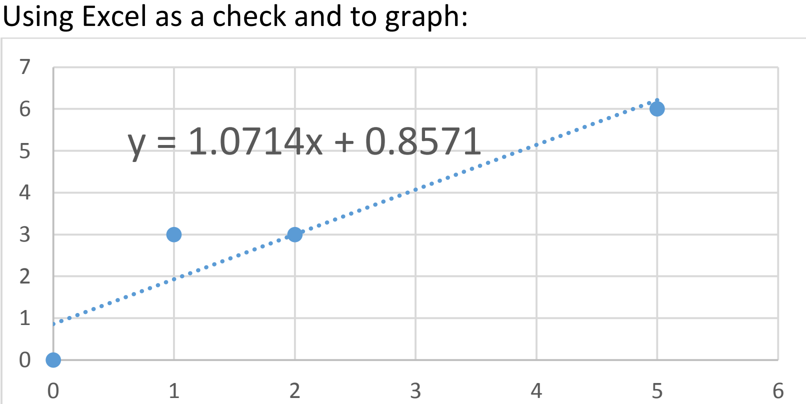
$$\begin{bmatrix}4 & 8 & | & 12\\ 8 & 30 & | & 39\end{bmatrix}\to\begin{bmatrix}1 & 2 & | & 3\\ 8 & 30 & | & 39\end{bmatrix}\to\begin{bmatrix}1 & 2 & | & 3\\ 0 & 14 & | & 15\end{bmatrix}\to\begin{bmatrix}1 & 2 & | & 3\\ 0 & 1 & | & 15/14\end{bmatrix}\to\begin{bmatrix}1 & 0 & | & 6/7\\ 0 & 1 & | & 15/14\end{bmatrix}$$

Hence linear equation is y=6/7+15/14x. Find r:

$$r=b-Ax=\begin{bmatrix}0\\ 3\\ 3\\ 6\end{bmatrix}-\begin{bmatrix}1 & 0\\ 1 & 1\\ 1 & 2\\ 1 & 5\end{bmatrix}\begin{bmatrix}12\\ 15\end{bmatrix}/14=\left(\begin{bmatrix}0\\ 42\\ 42\\ 84\end{bmatrix}-\begin{bmatrix}12\\ 27\\ 42\\ 87\end{bmatrix}\right)/14=\begin{bmatrix}-12\\ 15\\ 0\\ -3\end{bmatrix}/14$$ and RMSE is

$$\sqrt{(-12)^2+(15)^2+(-3)^2}/(14*2)=\sqrt{144+225+9}/28=\sqrt{378}/28\approx .6944$$

Using Excel as a check and to graph:



x	y	ym	err^2
0	0	6/7	36/49
1	3	1 13/14	1 29/196
2	3	3	0
5	6	6 3/14	9/196
		RMSE	0.694365

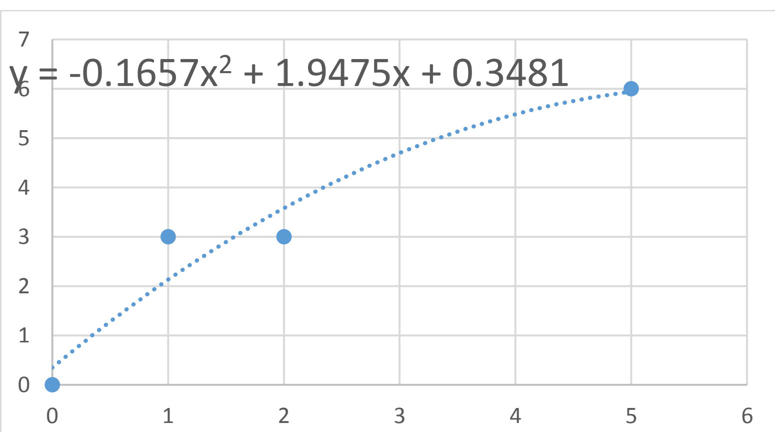
7. Find the best parabola through (0,0), (1,3), (2,3), (5,6), and find the RMSE. Graph the points as well as the solution. Find RMSE = $\sum_{i=1}^{4}(c_1+c_2t_i+c_3t_i^2-y_i)^2/\sqrt{4}$.

System is $[A|b]=\begin{bmatrix}1&0&0&|&0\\1&1&1&|&3\\1&2&4&|&3\\1&5&25&|&6\end{bmatrix}$. Normalize the system by multiplying by A^T:

$A^T[A|b]=\begin{bmatrix}1&1&1&1\\0&1&2&5\\0&1&4&25\end{bmatrix}\begin{bmatrix}1&0&0&|&0\\1&1&1&|&3\\1&2&4&|&3\\1&5&25&|&6\end{bmatrix}=\begin{bmatrix}4&8&30&|&12\\8&30&134&|&39\\30&134&642&|&165\end{bmatrix}$. As the numbers are too big

solve the resulting system using technology to get: [63/181 705/362 -30/181](see MATLAB file)
Hence linear equation is y=63/181 +705/362 t −30/181 t^2. Find r and RMSE using Excel:

x	y	ym	r	r^2
0	0	63/181	−63/181	0.1212
1	3	2 47/362	315/362	0.7572
2	3	3 105/181	−105/181	0.3365
5	6	5 341/362	21/362	0.0034
			RMSE	0.5519



Note that since the RMSE did not go down by much, this is not a great improvement over the linear model.