MAT 2630 Halleck Fall 2015 Practice Exam 1 v3

You will have 1 hour to do part 1 and 40 minutes to do part 2.

REMINDER: your 2 page (front and back) 1 sheet hand-written set of formulas and notes will be 10% of your grade. Make sure that you turn it in as part of the 2nd part of your exam.

Part 1, no MATLAB or computer is allowed beyond a graphing calculator.

1. a) Rewrite the polynomial in nested form:

$$P(x) = -3x^4 - 6x^3 + 18x^2 + 12x$$

$$= \left(\left(\left(-3x-6\right)x+18\right)x+12\right)x$$

b) In nested form evaluate at $x = -\frac{1}{2}$. Show 4 intermediate calculations and answer as a fraction.

$$P(-\frac{1}{2}) = \left(\left(\left(-3\left(-\frac{1}{2}\right)-6\right)\left(-\frac{1}{2}\right)+18\right)\left(-\frac{1}{2}\right)+12\right)\left(-\frac{1}{2}\right) \to \frac{3}{2}-6=-\frac{9}{2}$$

$$\to \left(-\frac{9}{2}\right)\left(-\frac{1}{2}\right)+18=\frac{9}{4}+18=\frac{81}{4} \to \left(\frac{81}{4}\right)\left(-\frac{1}{2}\right)+12=\frac{15}{8} \to \left(\frac{15}{8}\right)\left(-\frac{1}{2}\right)=-\frac{15}{16}$$

2. How many additions and multiplications are required to evaluate the polynomial $Q(x) = 3x^6 - 3x^4 + 9x^2 + 1$ in nested form by thinking of

a) Q(x) as a polynomial in x: Using a standard computer program which would perform additions even if they involve 0, there is 1 mult and 1 add for each degree so 6 mults and 6 adds in total.

b) Q(x) as a polynomial in x^2: Q(x)=P(x^2) where $P(x) = 3x^3 - 3x^2 + 9x + 1$. Evaluating x^2 is one mult. Evaluating P(x^2) is 3 mults and 3 adds so 4 mults and 3 adds total.

3. Find binary representations of the base 10 numbers (you don't need to convert to floating point)

a) 93 Create a table. Record whether dividend is even or odd then divide by 2 (int part). Although this is a palindrome, should be read backwards. As a check

$$1\times 64 + 0\times 32 + 1\times 16 + 1\times 8 + 1\times 4 + 0\times 2 + 1$$

$$= 64 + 16 + 8 + 4 + 1 = 93$$

b) 11.2 Read backwards the last 4 columns in the chart above for 11: 1011

For .2, multiply repeatedly the fractional part by 2 and remove the integer part, stopping if it's repeated.

Present number	0.2	0.4	0.8	1.6	1.2	0.4
Integer part		0	0	1	1	
Fractional part		0.4	0.8	0.6	0.2	

So answer is $1011.\overline{0011}$

4. Convert the base 10 numbers to binary and express as a binary floating point number fl(x) by using Rounding to Nearest Rule (but not as the machine represented number): a) 9.5

dividend	9	4	2	1
Odd or even?	1	0	0	1

a) .5 is trivially .1 in binary so answer is 1001.1=1.0001x2^3=1.0001x2^101

Present number	0.6	1.2	0.4	0.8	1.6	1.2
Integer part		1	0	0	1	
Fractional part		0.2	0.4	0.8	0.6	

b) $9.6 = 1001.\overline{1001} = 1.\overline{0011}\times 2\wedge 3 = 1.\overline{0011}\times 2\wedge 101$ Double only allows a mantissa of length 52 so there will only be 13 repeats of 0011 and since the 53rd bit is 0, it stays as is (1 is not added to the 52nd digit).

5. Do the following sums using IEEE Rounding to Nearest Rule. Express as a binary floating point number fl(x)

a) Before 1 is subtracted (1+(2^-51+2^-52+2^-54))-1, 53rd bit in mantissa is 0, so 1 not added to 52nd bit, so ans is

dividend	51	25	12	6	3	1
Odd or even?	1	1	0	0	1	1

b) (1+(2^-51+2^-52+2^-54+2^-60))-1 same work and answer as a)

6. Use Gaussian elimination to solve the 3x3 system. Describe the elementary row operation (or operations) used at each step. After the system is in triangular form, keep track of the arithmetic operations performed and verify that they total 9.

$$\begin{aligned} x+2y-z&=2 \\ y+5z&=6 \\ 3y+z&=4 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 3 & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & -14 & -14 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

	R3->R3-3R2	R3->R3/-14	R2->R2-5R3 R1->R1+R3	R1->R1+R3
divs		1	2	1
mults			2	1
adds			2	1

Which is 10, not 9 (not sure where I am off). Basic idea as explained in the book is that if a particular location will have a known result (like 1 or 0) then we do not have to do the operation(s) which lead to it. So essentially the divisions are used to compute the multipliers which are coming from working with the pivot and the row above the pivot. The multiplications and adds are working just with the constant column.

In fact, as I mentioned in class, I was not getting what I thought was the right answer using the code fragments in the book for section 1.1. What I was ignoring was that the resulting matrix is not the one that we get above in step 2. In fact, it is just the upper triangular portion. The lower triangular portion we know will all be 0's so from an efficiency standpoint, why bother computing those entries.

7. Suppose that your computer takes 0.005 seconds to perform back substitution for a system whose matrix is 5000x5000.
   a) How long will it take approximately do Gaussian elimination for this system?

$$\text{time for BS} = .005 = kn^2 = k(5000)^2 \Rightarrow k = .005/(5000)^2$$
$$\begin{aligned} \text{time for GE} &= (2/3)kn^3 = (2/3)\left(.005/(5000)^2\right)(5000)^3 \\ &= (2/3).005(5000) = 50/3 \end{aligned}$$

So it will take about 17 seconds to do GE

   b) By doing the most minimum of calculations, determine approximately how long it will take perform back substitution on this same computer for a system that is represented by a matrix of size 10000x10000.

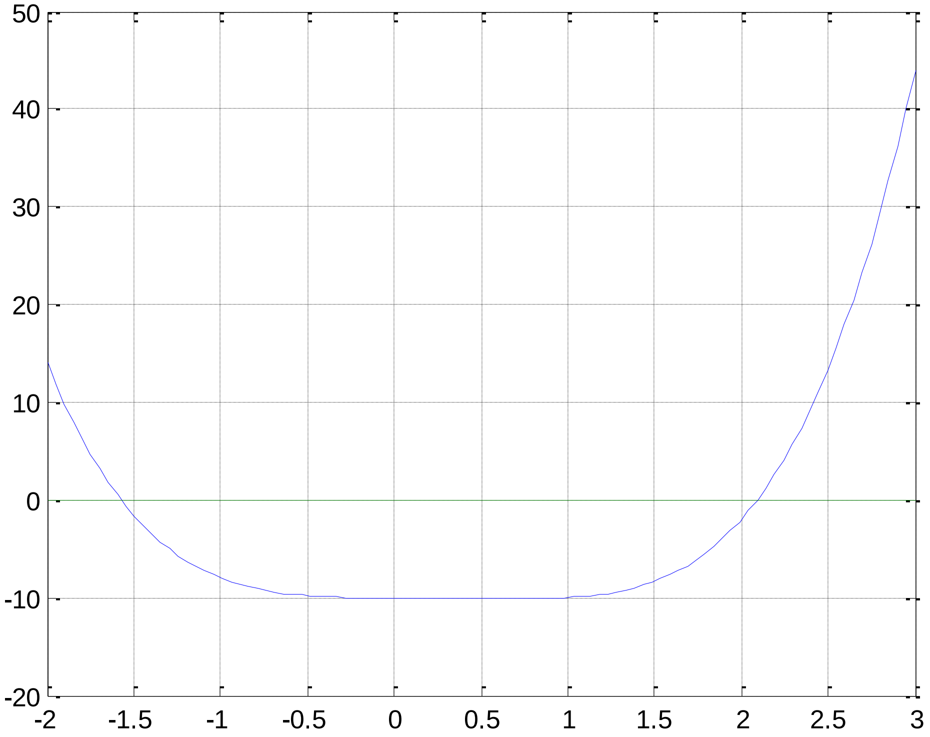
   Since the time is proportional to n^2, by doubling the size of n, the time will increase by a factor of 4.

   c) By doing the most minimum of calculations, determine approximately how long it will take to perform Gaussian elimination for a system that is represented by a matrix of size 10000x10000.

   Since the time is proportional to n^3, by doubling the size of n, the time will increase by a factor of 8.

Part 2 Using MATLAB (you may use templates that you create from a USB flash drive). Please turn in any code you use to Blackboard at the end of the period AS A SINGLE script m-FILE.

8. As a preliminary, bring all terms $x^4 = x^3 + 10$ to one side and graph the resulting function from x = −2 to x = 3. Provide a quick sketch on the axes below and label any zeros. $f(x) = x^4 - x^3 - 10$

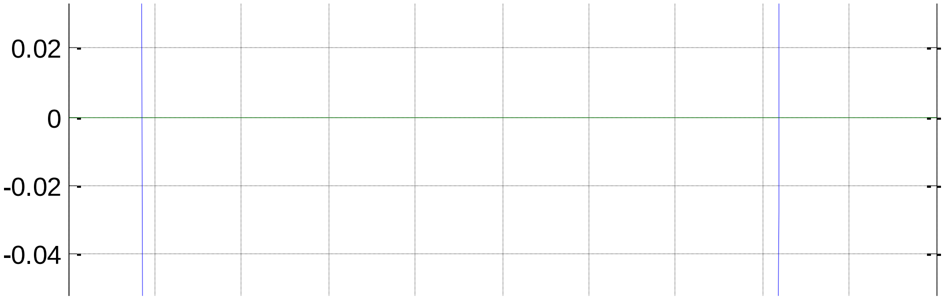


Here is the code for above:

```
x=linspace(-2,3);
plot(x,x.^4-x.^3-10,x,zeros(1,length(x)))
grid on
```

We want to zoom in on the x-axis to see if there are no multiple roots (only fragment of pic is shown).

```
axis([-2,3,-.1,.1])
```



a) State the IVT.
IVT says that given a continuous function over an interval [a,b] such that sign(f(a))*sign(f(b))=-1, then there exists c in [a,b] such that f(c)=0.

b) Using the IVT as a tool, find an interval of length 1 that contains the negative root of $x^4 = x^3 + 10$.

From graph, clear that f(-2) is pos and f(-1) is neg so [-2,-1] contains the neg root

c) Using the IVT as a tool, find an interval of length 1 that contains the positive root of $x^4 = x^3 + 10$.

From graph, clear that f(1.5) is neg and f(2.5) is pos so [1.5,2.5] contains the neg root

d) How many iterations are approximately required using the bisection method to find a solution to within 10^{-10} of the actual root?
It takes log2(10) bits to get one decimal place so we need 10(the exponent)* log2(10) bits. Using MATLAB:

```
10*log2(10)
```

we get 33.2 which we should round up to 34 to be GUARANTEED although the problem did say approximately.

9. Take the equation $x^4 = x^3 + 10$ and solve for x by subtracting x^3 and factoring out x^3 from the left side, dividing both sides by the binomial factor and taking the cube root. Would Fixed-point iteration converge given that the root is approximately 2.1? If so, would its convergence be better or worse than bisection?

$$\begin{aligned} x^4 &= x^3 + 10 \\ x^4 - x^3 &= 10 \\ x^3(x-1) &= 10 \\ x &= \sqrt[3]{10/(x-1)} = \sqrt[3]{10}(x-1)^{-1/3} \end{aligned}$$

taking the derivative and evaluating at 2.1

$$\begin{aligned} f(x) &= \sqrt[3]{10}(x-1)^{-1/3} \\ f'(x) &= -\left(\sqrt[3]{10}/3\right)(x-1)^{-4/3} \\ \left|f'(2.1)\right| &= \left|-\left(\sqrt[3]{10}/3\right)(2.1-1)^{-4/3}\right| \\ &= \sqrt[3]{10/1.1^4}/3 \end{aligned}$$

Using MATLAB

```
(10/1.1^4)^(1/3)/3
```

to evaluate we get: 0.63 which is more than ½ but less than 1 which means that converges linearly but more slowly than bisection (you should get about 5 additional bits for each 8 iterations).

10. We will now discuss Newton's method to find a root for $x^4 = x^3 + 10$.

a) Apply 2 steps of Newton's method with initial guess $x_0 = 2$. Don't use any of the m-functions that have been provided for this purpose. Instead use MATLAB essentially as a calculator. Show all the calculations.

$$\begin{aligned} f(x) &= x^4 - x^3 - 10 \\ f'(x) &= 4x^3 - 3x^2 \end{aligned}$$ Formula is $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$ so $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \dfrac{f(2)}{f'(2)} = 2 - \dfrac{-2}{20} = 2.1$

Which is accurate to 1 decimal place

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.092143277063912 \text{ (using MATLAB)}$$

```
f=@(x) x^4-x^3-10;
fp=@(x) 4*x^3-3*x^2;
2-f(2)/fp(2)
2.1-f(2.1)/fp(2.1
```

This agrees with the root (using MATLAB's builtin function) to 3 decimal places (10 bits)

```
roots([1 1 0 0 -10])
```

-2.092090469242937

-0.240140926665269 + 1.726923938917341i

-0.240140926665269 - 1.726923938917341i

1.572372322573478

b) How fast will Newton's method converge? Justify your answer.
As evidenced from above, the convergences is quadratic for the fractional bits (step 1: 3 bits, step 2: 10 bits). From the graph we can see that the both the real roots are simple. If we had a root with multiplicity, Newton's method would only provide linear convergence.