Exponential expressions

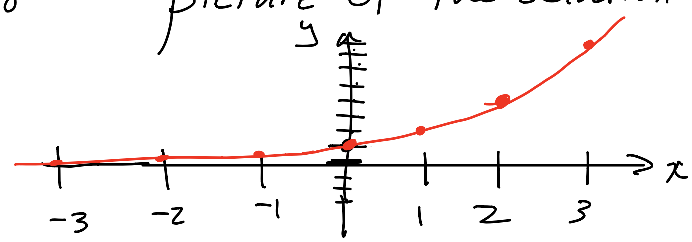
Consider the expression 2^x. Note the variable is in the exponent (as opposed to x^2 where the variable is the base).

What are the solutions to the equation $y = 2^x$? As always we look for an ordered pair of numbers (a, b) so that $b = 2^a$.

We see, for example, $(1,1)$ is not a solution since $1 \neq 2^1$. But $(0,1)$ is a solution since $1 = 2^0$. Other solutions can be seen by substituting various values in for x and finding y.

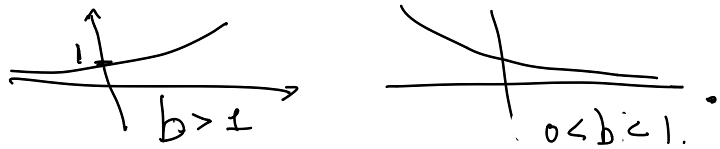
x	y
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

Graphing these solutions and drawing a smooth curve connecting them gives a picture of the solutions:



(Just like the stretch factor we can see the graph of $y = 4^x = 2^{2x}$ is compressed by a factor of $\frac{1}{2}$ along the x-axis — which is equivalently an adjustment of the scale of the x-axis: replacing $1, 2, 3, \ldots$ etc by $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots$)

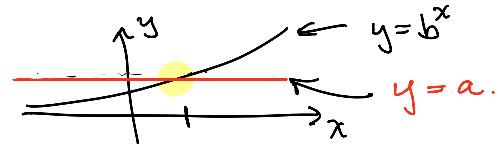
In general, a sketch of $y = b^x = \left(\frac{1}{b}\right)^{-x}$, $\frac{1}{b} > 1$.
looks like



We define the expression $\log_b a$ to be the solution to the equation $b^x = a$. b is called the base.

Graphically, we look at the solution to

$$\begin{cases} y = b^x \\ y = a \end{cases} :$$



The x-coordinate of the intersection is called $\log_b a$. Note there is exactly <u>one</u> solution if $a > 0$.

<u>Example</u>: $\log_2 8$ is the solution to $2^x = 8$.

But we know $2^3 = 8$ so, $\log_2 8 = 3$.

<u>Example</u>: $\log_5\left(\frac{1}{25}\right)$ is the solution to $5^x = \frac{1}{25}$, but $5^x = 5^{-2}$ so $x = -2$ is a solution. So

$$\log_5 \frac{1}{25} = -2.$$

<u>Example</u> $\log_b 1 = 0$ since $b^0 = 1$.

<u>Example</u> $\log_{10} 5$ is the solution to $10^x = 5$. The calulator

$\boxed{\log = \log_{10}}$

Can solve this one by using the $\boxed{\log}$ button. Note the similarity of the positive solution of $x^2 = 5$ on the calculator using $\boxed{\sqrt{\ }}$.

$\boxed{\log}$ and $\boxed{10^x}$ share a button and $\boxed{x^2}$ and $\boxed{\sqrt{\ }}$ share a button. $\boxed{\ln}$ and $\boxed{e^x}$ also share a button. $\ln a := \log_e a$.

We see $\log_{10} 5 \approx 0.699$.

Example: $\ln e^2 = 2$ since $e^x = e^2$ has 2 as its solution.

Properties of exponents	Corresponding Properties of logs
① $b^0 = 1$	$\log_b 1 = 0$
② $b^x b^y = b^{x+y}$ Let $u = b^x, v = b^y$ then $uv = b^{x+y}$	$\log_b u \cdot v = \log_b u + \log_b v$ $(x + y = x + y)$
③ $(b^x)^r = b^{xr}$ $u = b^x$ $(\log_b u = x)$	$\log_b u^r = r \log_b u$ $(xr = rx)$

$u^r = (b^x)^r \overset{③}{=} b^{xr} \Rightarrow \log_b u^r = xr$

(4) Change of base formula

$$r = r$$

$$\iff a^{\log_a r} = b^{\log_b r}$$

$$= \left(a^{\log_a b}\right)^{\log_b r}$$

$$= a^{\log_a b \cdot \log_b r}$$

$$\iff \log_a r = \log_a b \cdot \log_b r$$

$$\iff \log_b r = \frac{\log_a r}{\log_a b}$$

base b   base a

Summary:

(1) $\log_b 1 = 0$

(2) a) $\log_b(xy) = \log_b x + \log_b y$

b) $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_a y$

uses 2a with (3) $(r = -1)$

(3) $\log_b x^r = r \log_b x$

(4) $\log_b r = \dfrac{\log_a r}{\log_a b}$.

Examples:

(1) $\log_5 7 = \dfrac{\log_{10} 7}{\log_{10} 5} \approx 1.209$

calculator

2) $$\log_b \sqrt{xy}\, z^2 \overset{2}{=} \log_b (xy)^{1/2} + \log_b z^2$$

$$\overset{3}{=} \frac{1}{2}\log_b xy + 2\log_b z$$

$$\overset{2}{=} \frac{1}{2}\left(\log_b x + \log_b y\right) + 2\log_b z$$

3) $$\log_7\left(\frac{x^2 y}{z w^2}\right)$$

$$\overset{2}{=} \log_7 x^2 + \log_7 y - \log_7 z - \log_7 w^2$$

$$= 2\log_7 x + \log_7 y - \log_7 z - 2\log_7 w$$

4) $$\log_2 x - 2\log_2 y + \frac{1}{2}\left(\log_2 z - \log_2 w\right) - 1$$

$$\overset{3}{=} \log_2 x - \log_2 y^2 + \log_2 z^{1/2} - \log_2 w^{1/2} - \log_2 2$$

$$\overset{2}{=} \log_2\left(\frac{x z^{1/2}}{2 y^2 w^{1/2}}\right)$$

$$= \log_2\left(\frac{x}{2y^2}\sqrt{\frac{z}{w}}\right).$$