Magnetostatics introduction

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Magnetostatics deals with the study of the magnetic fields due to steady (i.e. time-independent) currents. If the charge density per unit volume is zero, the electric field will also be zero. Only the magnetic field will be part of the problem. We already saw that the two equations that describe the magnetic field in a magnetostatics problem are

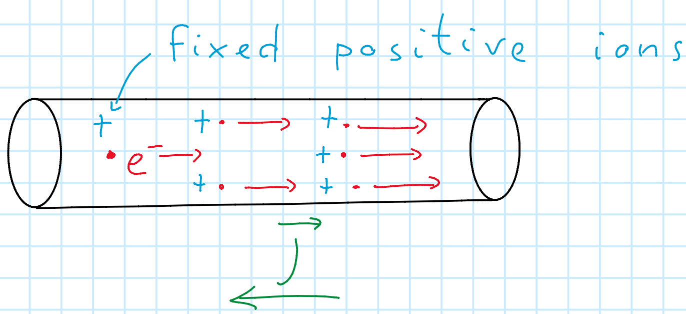
$$\boxed{\nabla \times \bar{B} = \mu_0 \bar{J}} \quad , \quad \boxed{\nabla \cdot \bar{B} = 0}$$

If one fixes the current density j, the equation above have a unique solution. Our goal is to find that solution.
In addition, notice that if E =0 then B is time-independent

$$\nabla \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \quad \text{and} \quad \bar{E} = 0 \implies \frac{\partial \bar{B}}{\partial t} = 0$$

Steady currents

Notice that in a conducting wire we are precisely in the case in which the charge density per unit volume is zero but the current density is not zero. Indeed in a wire the electrons move in the lattice of the positive ions in the metal, the overall charge density remains zero



The charge density in each section of the wire remains zero $\longrightarrow$ $\rho = 0$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0 \longrightarrow \nabla \cdot \bar{J} = 0$$

This is in agreement with Ampere's law

$$0 = \underbrace{\nabla \cdot (\nabla \times \bar{B})}_{\nabla \cdot (\nabla \times \bar{v}) = 0 \quad \text{always!}} = \mu_0 \nabla \cdot \bar{J}$$

More formally, one can start from Ampere-Maxwell's law and see what happens when E = 0 (or even simply E is time independent)

$$\nabla \times \bar{B} - \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} = \mu_0 \bar{J} \longrightarrow \nabla \times \bar{B} = \mu_0 \bar{J}$$

By taking the time derivative of the equation above

$$\frac{\partial}{\partial t} (\nabla \times \bar{B}) = \nabla \times \frac{\partial \bar{B}}{\partial t} = 0 = \frac{\partial}{\partial t} (\mu_0 \bar{J}) = \mu_0 \frac{\partial \bar{J}}{\partial t}$$

Vector potential

We already observed that, since $\nabla \cdot \bar{B} = 0$
One can always write

$$\bar{B} = \nabla \times \bar{A} \quad \text{vector potential}$$

Ampere's law implies then that

$$\nabla \times \bar{B} = \nabla \times (\nabla \times \bar{A}) = - \Delta \bar{A} + \nabla (\nabla \cdot \bar{A}) = \mu_0 \bar{J}$$

Several different vector potentials describe the same magnetic field, indeed, suppose that

$$\bar{A}' = \bar{A} + \nabla \chi \longrightarrow \bar{B} = \nabla \times \bar{A}' = \nabla \times \bar{A}$$

since $$\nabla \times (\nabla \chi) = 0 \quad \text{always}$$

Two vector potentials which differ by a gradient are said to be connected by a gauge transformation. All vector potentials which are connected by gauge transformations are equivalent, i.e. they describe the same magnetic field.

One can therefore impose an additional condition which must be satisfied by the vector potential (this operation is called "choosing the gauge"). In magnetostatics it is convenient to work in Coulomb's gauge

$$\nabla \cdot \bar{A} = 0$$

Theorem: It is always possible to find a gauge transformation from a generic vector potential to a vector potential which satisfies Coulomb's gauge condition.

Proof: Consider a generic vector potential which generates the correct magnetic field B for a given problem. One can calculate the divergence of this potential, this divergence will be a scalar function

$$\nabla \cdot \bar{A} = \psi(\bar{x})$$

One can then consider a gauge transformation

$$\bar{A}' = \bar{A} + \nabla \chi$$

$$\hookrightarrow \quad \nabla \cdot \bar{A}' = \nabla \cdot \bar{A} + \nabla \cdot \nabla \chi = \psi + \Delta \chi$$

One can then require that the new vector potential satisfies the Coulomb's gauge condition

$$\nabla \cdot \bar{A}' = 0 \longrightarrow \Delta \chi = -\psi$$

POISSON'S TYPE EQUATION

The Poisson's equation can be solved, and the appropriate gauge transformation found.