Torque on a magnetic dipole

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We saw that the force acting on a magnetic dipole is

$$\bar{F} = -\nabla U \quad \text{with} \quad U = -\bar{m}\cdot\bar{B}$$

$$\bar{F} = \nabla(\bar{m}\cdot\bar{B}) = \bar{m}\times(\nabla\times\bar{B}) + \cancelto{0}{\bar{B}\times(\nabla\times\bar{m})} + (\bar{m}\cdot\nabla)\bar{B}$$

$$+ \cancelto{0}{(\bar{B}\cdot\nabla)\bar{m}}$$

$$= \bar{m}\times(\underbrace{\nabla\times\bar{B}}_{=0 \text{ if } \bar{J}=0}) + (\bar{m}\cdot\nabla)\bar{B}$$

The force is then zero if B is constant.

Here we want to determine the torque acting on the magnetic dipole. One can start from the vector potential due to a dipole placed at x_0.

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\frac{\bar{m}\times(\bar{x}-\bar{x}_0)}{|\bar{x}-\bar{x}_0|^3} \longrightarrow \bar{B} = \nabla\times\bar{A}$$

rem

$$\nabla\times(\bar{a}\times\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b} + \bar{a}(\nabla\cdot\bar{b}) - \bar{b}(\nabla\cdot\bar{a})$$

$$\nabla\times\left(\bar{m}\times\frac{(\bar{x}-\bar{x}_0)}{|\bar{x}-\bar{x}_0|^3}\right) = -(\bar{m}\cdot\nabla)\left(\frac{\bar{x}-\bar{x}_0}{|\bar{x}-\bar{x}_0|^3}\right) + \bar{m}\,\nabla\cdot\left(\frac{\bar{x}-\bar{x}_0}{|\bar{x}-\bar{x}_0|^3}\right)$$

$$\bar{B}(\bar{x}) = \frac{\mu_0}{4\pi}\left[\bar{m}\,\nabla\cdot\frac{\bar{x}-\bar{x}_0}{|\bar{x}-\bar{x}_0|^3} - (\bar{m}\cdot\nabla)\frac{\bar{x}-\bar{x}_0}{|\bar{x}-\bar{x}_0|^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[-\bar{m}\,\nabla\cdot\left(\nabla\frac{1}{|\bar{x}-\bar{x}_0|}\right) - (\bar{m}\cdot\nabla)\frac{\bar{x}-\bar{x}_0}{|\bar{x}-\bar{x}_0|^3}\right]$$

rem

$$\nabla(\bar{a}\cdot\bar{b}) = \bar{a}\cdot\nabla\bar{b} + \bar{b}\cdot\nabla\bar{a}$$

$$\bar{B}(\bar{x}) = \frac{\mu_0}{4\pi}\left[-\bar{m}\,\Delta\frac{1}{|\bar{x}-\bar{x}_0|} - \nabla\left(\frac{\bar{m}\cdot(\bar{x}-\bar{x}_0)}{|\bar{x}-\bar{x}_0|^3}\right)\right]$$

From Ampere's equation in magnetostatics one has

$$\nabla \times \bar{B} = \mu_0 \bar{J} \longrightarrow \bar{J} = \frac{1}{\mu_0} \nabla \times \bar{B}$$

$$\bar{J}(\bar{x}) = \frac{1}{4\pi}\left[-\nabla \times \left(\bar{m} \underbrace{\Delta \frac{1}{|\bar{x}-\bar{x}_0|}}_{-4\pi\,\delta^{(3)}(\bar{x}-\bar{x}_0)} \right) - \underbrace{\nabla \times \nabla}_{=0} \frac{\bar{m}\cdot(\bar{x}-\bar{x}_0)}{|\bar{x}-\bar{x}_0|^3} \right]$$

$$= \nabla \times \left(\bar{m}\, \delta^{(3)}(\bar{x}-\bar{x}_0) \right)$$

rem $\quad \nabla \times (f \bar{A}) = f\, \nabla \times \bar{A} - \bar{A} \times \nabla f$

$$\bar{J}(\bar{x}) = -\bar{m} \times \nabla \delta^{(3)}(\bar{x}-\bar{x}_0)$$

CURRENT DENSITY OF A MAGNETIC DIPOLE

The torque that we want to evaluate is

$$\bar{\tau} = \int \bar{x} \times d\bar{F} = \int d^3x\; \bar{x} \times (\bar{J} \times \bar{B})$$

external B, not the B due to $\bar{J}$

$$= -\int d^3x\; \bar{x} \times \left[\left(\bar{m} \times \nabla \delta^{(3)}(\bar{x}-\bar{x}_0) \right) \times \bar{B}(\bar{x}) \right]$$

$$= \int d^3x\; \bar{x} \times \left[\bar{B}(\bar{x}) \times \left(\bar{m} \times \nabla \delta^{(3)}(\bar{x}-\bar{x}_0) \right) \right]$$

rem

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\,\bar{b} - (\bar{a}\cdot\bar{b})\,\bar{c}$$

$$\bar{\tau} = \int d^3x\; \bar{x} \times \left[\left(\bar{B} \cdot \nabla \delta^{(3)}(\bar{x}-\bar{x}_0) \right) \bar{m} - \left(\bar{B}\cdot\bar{m} \right) \nabla \delta^{(3)}(\bar{x}-\bar{x}_0) \right]$$

Consider now a component of the torque

$$\tau_i = \int d^3x\; \varepsilon_{ijk}\, x_j \left[B_\ell\, \partial_\ell\, \delta^{(3)}(\bar{x}-\bar{x}_0)\, m_k - m_\ell B_\ell\, \partial_k \delta^{(3)}(\bar{x}-\bar{x}_0) \right]$$

$$= -\int d^3x\; \varepsilon_{ijk}\, \partial_\ell (x_j B_\ell)\, \delta^{(3)}(\bar{x}-\bar{x}_0)\, m_k$$

$$+ \int d^3x\; \varepsilon_{ijk}\, \partial_k (x_j B_\ell)\, \delta^{(3)}(\bar{x}-\bar{x}_0)\, m_\ell$$

BY PARTS

In the integration by parts, the surface terms vanish since we are integrating over R^3 and the delta function is zero everywhere except for x_0.

$$\tau_i = \int d^3x \; \delta^{(3)}(\bar{x} - \bar{x}_0) \; \varepsilon_{ijk} \left[m_\ell \, \partial_k - m_k \, \partial_\ell \right] x_j B_\ell$$

$$= \int d^3x \; \delta^{(3)}(\bar{x} - \bar{x}_0) \; \varepsilon_{ijk} \Big[\underbrace{\cancel{m_\ell \, \delta_{jk} B_\ell}}_{\to 0,\ \varepsilon_{ijk}\delta_{jk} = 0} + m_\ell \, x_j \, \partial_k B_\ell$$

$$- m_k \, \delta_{\ell j} \, B_\ell - m_k \, x_j \, \underbrace{\cancel{\partial_\ell B_\ell}}_{\nabla \cdot \bar{B} = 0} \Big]$$

$$= \int d^3x \; \delta^{(3)}(\bar{x} - \bar{x}_0) \; \varepsilon_{ijk} \, x_j \, \overbrace{(\bar{m} \cdot \partial_k \bar{B})}^{\partial_k (\bar{m}\cdot\bar{B})} - \int d^3x \; \delta^{(3)}(\bar{x} - \bar{x}_0) \; \varepsilon_{ijk} \, m_k B_j$$

$$= \left[\bar{m} \times \bar{B}(\bar{x}_0) \right]_i + \left[\bar{x}_0 \times \nabla (\bar{m} \cdot \bar{B}) \right]_i$$

We saw that the force acting on a dipole is

$$\bar{F} = -\nabla \left(-\bar{m} \cdot \bar{B} \right) = \nabla \left(\bar{m} \cdot \bar{B} \right)$$

$$\bar{x}_0 \times \nabla (\bar{m} \cdot \bar{B}) = \bar{x}_0 \times \bar{F}(\bar{x}_0)$$

So that finally

$$\bar{\tau} = \bar{m} \times \bar{B}(\bar{x}_0) + \bar{x}_0 \times \bar{F}(\bar{x}_0)$$

Consequently if the dipole is located at the origin of the frame of reference one finds

$$\bar{x}_0 = 0 \longrightarrow \boxed{\bar{\tau} = \bar{m} \times \bar{B}}$$

TORQUE ACTING ON A MAGNETIC DIPOLE