Magnetic Dipoles

Monday, January 21, 2019 10:14 AM

Here we want to show that the magnetic field far away from a loop of current fall off as B ~ 1/r^3. This behavior of the field is called dipole-like in analogy with what we found for the electric field. It is useful to start by analyzing a circular loop of current.

Circular Current Loop

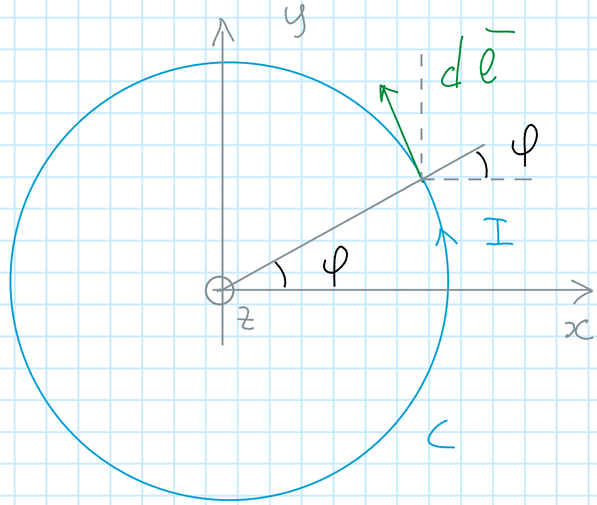
Let consider a circular loop of radius a lying on the x-y plane.

$$\bar{j}(\bar{r}') = \frac{I\, d\bar{\ell}}{a^2}\, \delta(r-a)\, \delta\left(\theta - \frac{\pi}{2}\right)$$

(observe that

$$\int_0^{\pi} d\theta \int_0^{\infty} dr\; r\, j(r) = I$$

)



$$\frac{d\bar{\ell}}{a} = \frac{dx}{a}\hat{\imath} + \frac{dy}{a}\hat{\jmath} = d\varphi\, \hat{\varphi} = d\varphi\left(-\sin\varphi\, \hat{\imath} + \cos\varphi\, \hat{\jmath}\right)$$

One can then calculate the vector potential

$$\bar{A}(\bar{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\bar{\ell}'}{|\bar{r}-\bar{r}'|}$$

rem $\bar{r}'$ is on the loop, $\bar{r}$ is a generic point in space

We now assume to be very far away from the loop, so that $r \gg r'$. One can then expand the denominator in the integrand

$$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{r} + \sum_{i=1}^{3} x_i' \frac{\partial}{\partial x_i'} \frac{1}{|\bar{r}-\bar{r}'|} + \dots$$

$$= \frac{1}{r} + \bar{r}' \cdot \nabla' \frac{1}{|\bar{r}-\bar{r}'|}\Bigg|_{\bar{r}'=0} + \dots$$

$$= \frac{1}{r} + \bar{r}' \cdot \left(+ \frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3} \right)\Bigg|_{\bar{r}'=0} + \dots$$

$$= \frac{1}{r} + \frac{\bar{r}' \cdot \bar{r}}{r^3} + \dots$$

One then finds

$$\bar{A}(\bar{r}) = \frac{\mu_0 I}{4\pi} \oint_C d\bar{\ell}' \left(\frac{1}{r} + \frac{\bar{r}' \cdot \bar{r}}{r^3} + \dots \right)$$

Now one can observe that

$$\oint_C \frac{d\bar{\ell}'}{a} = \int_0^{2\pi} d\varphi \, (-\sin\varphi)\, \hat{\imath} + \int_0^{2\pi} d\varphi \, \cos\varphi \, d\varphi = 0$$

Therefore the first term in the integral vanishes.

The first non vanishing integral is the second one

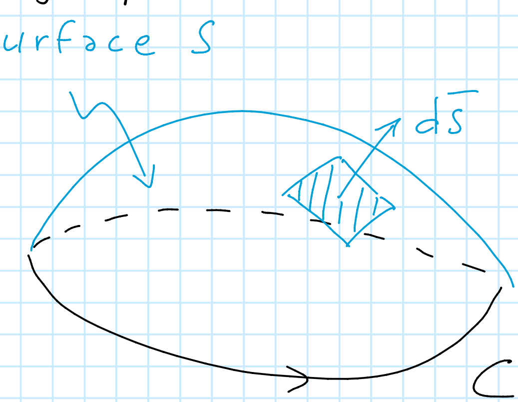
$$\bar{A}(\bar{r}) = \frac{\mu_0 I}{4\pi} \oint_C d\bar{\ell}' \, \frac{\bar{r}' \cdot \bar{r}}{r^3}$$

In order to calculate the integral above we first prove the following theorem (David Tong lectures)

Theorem

$$\oint_C d\bar{\ell}' \, (\bar{r} \cdot \bar{r}') = \left(\int_S d\bar{S} \right) \times \bar{r}$$

Where S a surface bound by the path C.



Proof

Consider the following dot product of the integral that we want to calculate with a generic constant vector v

$$\bar{v} \cdot \oint_C d\bar{\ell}' \, (\bar{r} \cdot \bar{r}') = \oint_C d\bar{\ell}' \cdot \underbrace{\left[\bar{v} \, (\bar{r} \cdot \bar{r}') \right]}_{\equiv \bar{K}}$$

$$= \oint_C d\bar{\ell}' \cdot \bar{K} = \int_S d\bar{S} \cdot (\nabla \times \bar{K}) \quad \text{(Stoke's theorem)}$$

$$\bar{v} \cdot \oint_C d\bar{\ell}' \, (\bar{r} \cdot \bar{r}') = \int_S d\bar{S} \cdot \left[\nabla' \times \left(\bar{v} (\bar{r} \cdot \bar{r}') \right) \right]$$

$$= \int_S dS_i \left[\varepsilon_{ijk} \, \partial'_j \left(v_k \, r_\ell \, r'_\ell \right) \right]$$

$$= \int_S dS_i \; \varepsilon_{ijk} \, v_k \, r_\ell \, \delta_{j\ell}$$

$$= \int_S dS_i \; \varepsilon_{ijk} \, v_k \, r_j = v_k \, \varepsilon_{kij} \int_S dS_i \, r_j$$

$$= \bar{v} \cdot \left[\left(\int_S d\bar{S} \right) \times \bar{r} \right]$$

No assumption was made on the nature of vector v, except that it is a constant vector, therefore one can conclude that

$$\oint_C d\bar{\ell}' \, (\bar{r} \cdot \bar{r}') = \left(\int_S d\bar{S} \right) \times \bar{r}$$

Q.E.D.

By applying the theorem above one then finds that

$$\bar{A}(\bar{r}) = \frac{\mu_0 I}{4\pi} \, \frac{1}{r^3} \underbrace{\left(\int_S d\bar{S} \right)}_{\equiv \, \bar{S}} \times \bar{r} = \frac{\mu_0}{4\pi} \, \frac{\bar{m} \times \bar{r}}{r^3}$$

$$\boxed{\bar{m} \equiv I \, \bar{S} = I \int_S d\bar{S}}$$

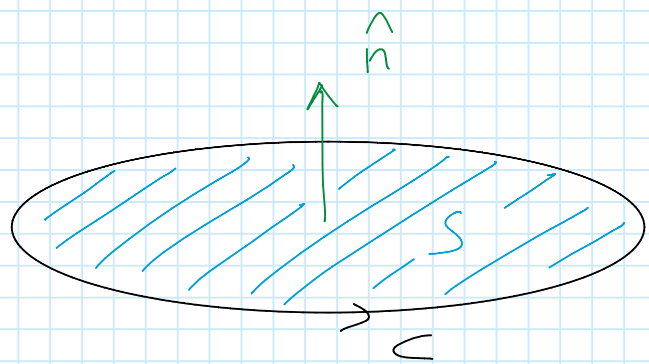
MAGNETIC DIPOLE MOMENT

Observe that one can choose any surface bound by C. One therefore can choose the flat surface bound by C. The direction of the S is then simply perpendicular to the flat surface and

unit vector $\perp$ to the surface

$$\bar{S} = \int_S d\bar{S} = S\,\hat{n}$$

flat area bound by C



Magnetic field

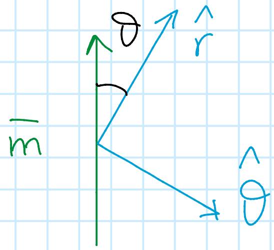
One can then calculate the magnetic field due to the dipole vector potential found above. Assume that m is in the direction of the z axis and use spherical coordinates

$$\bar{B}(\bar{r}) = \nabla \times \bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\left[-(\bar{m}\cdot\nabla)\frac{\bar{r}}{r^3} + \bar{m}\left(\nabla\cdot\frac{\bar{r}}{r^3}\right)\right]$$

rem

$$\left[\nabla\times(\bar{A}\times\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} - (\bar{A}\cdot\nabla)\bar{B} + \bar{A}(\nabla\cdot\bar{B}) - \bar{B}(\nabla\cdot\bar{A})\right]$$

$$\nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = \nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$$



$$\overline{m} = m\cos\vartheta\,\hat{r} - m\sin\vartheta\,\hat{\vartheta}$$

$$\overline{m}\cdot\nabla = m_z \frac{\partial}{\partial z}$$

$$\left(\overline{m}\cdot\nabla\right)\frac{\overline{r}}{r^3} =$$

$$= m\frac{\partial}{\partial z}\left(\frac{1}{r^2}\sin\vartheta\cos\varphi\,\hat{\imath} + \frac{1}{r^2}\sin\vartheta\sin\varphi\,\hat{\jmath} + \frac{1}{r^2}\cos\vartheta\,\hat{k}\right)$$

$$= m\frac{\partial}{\partial z}\left(\frac{x}{r^3}\hat{\imath} + \frac{y}{r^3}\hat{\jmath} + \frac{z}{r^3}\hat{k}\right)$$

$$= m\left(-3\frac{x}{r^4}\frac{\partial r}{\partial z}\hat{\imath} - 3\frac{y}{r^4}\frac{\partial r}{\partial z}\hat{\jmath} - 3\frac{z}{r^4}\frac{\partial r}{\partial z}\hat{k}\right.$$

$$\left. + \frac{1}{r^3}\hat{k}\right) = (*)$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{\frac{1}{2}} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} 2z$$

$$= \frac{z}{r}$$

$$(*) = \left[-3z\,\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{r^5} + \frac{1}{r^3}\hat{k}\right]m$$

$$= \left[-3\,\frac{\cos\vartheta}{r^3}\hat{r} + \frac{1}{r^3}\hat{k}\right]m$$

So that finally

$$\overline{B}(\overline{r}) = \mu_0\left[\;(\overline{m}\cdot\nabla)\,\overline{r}\;\right]$$

$$\bar{B}(\bar{r}) = \frac{\mu_0}{4\pi}\left[-(\bar{m}\cdot\nabla)\frac{\bar{r}}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{3\, m\cos\vartheta\, \hat{r}}{r^3} + \frac{m\hat{k}}{r^3}\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{3(\bar{m}\cdot\hat{r})\hat{r}}{r^3} + \frac{\bar{m}}{r^3}\right]$$

The magnetic field has the same functional form of the dipole electric field, which justifies the name of magnetic dipole for this current configuration.