Biot Savart Law

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One wants to determine the magnetic field due to a generic current density. We start by studying how to determine the vector potential for a generic current density. We already saw that

$$\nabla \times \bar{B} = -\Delta \bar{A} + \nabla(\underbrace{\nabla \cdot \bar{A}}_{=0 \text{ Coulomb's gauge}}) = \mu_0 \bar{J}$$

$$\boxed{\Delta \bar{A} = -\mu_0 \bar{J}}$$

Poisson type equation

Therefore the vector potential can be determined by solving a Poisson-like equation for each component of the vector potential. In cartesian coordinates

$$\Delta A_i = -\mu_0 j_i \qquad (i = 1, 2, 3)$$

We already encountered Poisson's equation in electrostatics

$$\Delta \varphi = -\frac{\rho}{\varepsilon_0}$$

The solution (in integral form) of the equation above is

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho(\bar{r}')}{|\bar{r} - \bar{r}'|}$$

Therefore, the solution of the Poisson's equation for the components of the vector potential will be

$$A_i(\bar{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{J_i(\bar{r}')}{|\bar{r} - \bar{r}'|}$$

$$\boxed{\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\bar{J}(\bar{r}')}{|\bar{r} - \bar{r}'|}}$$

The solution (in integral form) written above automatically satisfies Coulomb's gauge condition. In fact

$$\nabla \cdot \bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int_V d^3\bar{r}' \, \nabla \cdot \left(\frac{\bar{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \right)$$

derivative w.r.t $\bar{r}$

$$= \frac{\mu_0}{4\pi} \sum_{i=1}^{3} \int d^3r' \, \partial_i \frac{J_i(\bar{r}')}{|\bar{r} - \bar{r}'|}$$

$$= \frac{\mu_0}{4\pi} \int d^3r' \, J_i(\bar{r}') \frac{\partial}{\partial x_i} \frac{1}{|\bar{r} - \bar{r}'|} \qquad \left(i \text{ summed over} \right)$$

$$= -\frac{\mu_0}{4\pi} \int d^3r' \, J_i(\bar{r}') \frac{\partial}{\partial x_i'} \frac{1}{|\bar{r} - \bar{r}'|}$$

$$= -\frac{\mu_0}{4\pi} \int d^3r' \, \bar{J}(\bar{r}') \cdot \nabla' \frac{1}{|\bar{r} - \bar{r}'|}$$

One can now integrate by parts

$$\nabla \cdot \bar{A}(\bar{r}) = -\frac{\mu_0}{4\pi} \int_V d^3r' \left[\nabla' \cdot \left(\frac{\bar{J}(\bar{r}')}{|\bar{r} - \bar{r}'|} \right) - \underbrace{\left(\nabla' \cdot \bar{J}(\bar{r}') \right)}_{=0 \text{ from continuity equation}} \frac{1}{\bar{r} - \bar{r}'} \right]$$

$$= -\frac{\mu_0}{4\pi}\int_{\partial V} d\bar{S}' \cdot \frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|} = 0$$

The last equality applies if the current density is localized in some region of space, so that the current density vanishes on the surface of the volume considered.

Magnetic field

It is possible to derive an integral equation for the magnetic field starting from the expression for the vector potential

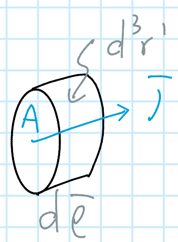
$$\bar{B}(\bar{r}) = \nabla \times \bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int_V d^3r' \left(-\bar{J}(\bar{r}') \times \underbrace{\nabla \frac{1}{|\bar{r}-\bar{r}'|}}_{= -\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}} \right)$$

$$\boxed{\bar{B}(\bar{r}) = \frac{\mu_0}{4\pi}\int_V d^3r' \, \frac{\bar{J}(\bar{r}') \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}}$$

There is a special case of the equation above which is often useful in problems. Consider a current which flows in a thin wire. Furthermore, let's name C the curve described by the thin wire.

$$\bar{J}(\bar{r}') \, d^3r' = j \, A \, d\bar{\ell}$$

assume j constant in magnitude



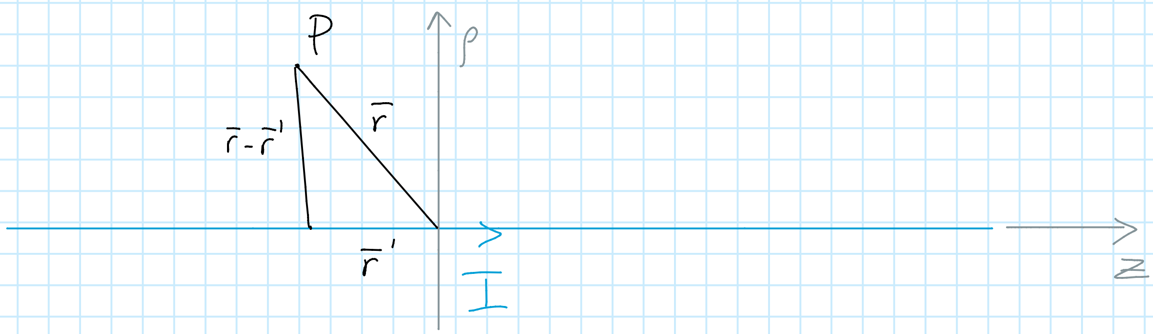
$$\boxed{jA \equiv I}$$

$$\boxed{\bar{B}(\bar{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\bar{\ell}' \times (\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}}$$

BIOT SAVART LAW

Straight wire

One can reconsider the case of a straight wire by using Biot Savart Law. It is convenient to use cylindrical coordinates with the z axis aligned along the straight wire.



$$\bar{r}' = z'\hat{k} \qquad \bar{r} = \rho\hat{\rho} + z\hat{k} \qquad d\bar{\ell}' = dz'\hat{k}$$

$$\bar{r}-\bar{r}' = \rho\hat{\rho} + (z-z')\hat{k}$$

$$d\bar{\ell}' \times (\bar{r}-\bar{r}') = \rho\, dz'\, \hat{k}\times\hat{\rho} = \rho\, dz'\, \hat{\varphi}$$

$$\bar{B}(\bar{r}) = \frac{\mu_0 I}{4\pi}\hat{\varphi} \underbrace{\int_{-\infty}^{+\infty} du \frac{\rho}{(\rho^2+u^2)^{\frac{3}{2}}}}_{=\frac{2}{\rho}} = \frac{\mu_0 I}{2\pi\rho}\hat{\varphi}$$

one can replace $z'-z = u$

$$\int_{-\infty}^{+\infty} du \; \frac{\rho}{(\rho^2 + u^2)^{\frac{3}{2}}} = \int_{-\infty}^{+\infty} du \; \frac{1}{\rho^2} \; \frac{1}{\left(1 + \frac{u^2}{\rho^2}\right)^{\frac{3}{2}}}$$

$u' \equiv \frac{u}{\rho}$

$$= \frac{1}{\rho} \int_{-\infty}^{+\infty} du \; \frac{1}{(1 + u^2)^{\frac{3}{2}}} = \frac{2}{\rho} \int_{0}^{\infty} du \; \frac{1}{(1 + u^2)^{\frac{3}{2}}}$$

$$I = \int_{0}^{\infty} du \; (1 + u^2)^{-\frac{3}{2}}$$

$u \equiv \tan \vartheta$

$u = 0 \rightarrow \vartheta = 0$

$$du = \frac{d\vartheta}{\cos^2 \vartheta} \qquad 1 + u^2 = \frac{1}{\cos^2 \vartheta}$$

$u = \infty \rightarrow \vartheta = \frac{\pi}{2}$

$$I = \int_{0}^{\frac{\pi}{2}} \frac{d\vartheta}{\cos^2 \vartheta} \cos^3 \vartheta = \int_{0}^{\frac{\pi}{2}} d\vartheta \; \cos \vartheta = \sin \vartheta \Big|_{0}^{\frac{\pi}{2}} = 1$$

Q.E.D.