Boundary value problems with azimuthal symmetry

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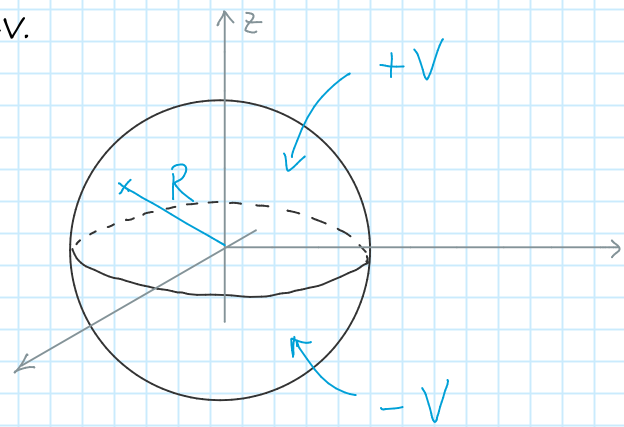
We showed that the potential in a problem with azimuthal symmetry is of the general form

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\vartheta)$$

In addition, we pointed out that the coefficients B are zero if we are inside a sphere and the potential must be finite inside the sphere, and the coefficients A are zero if we consider the space outside a sphere and the potential should vanish at infinity.

Problem 1

Find the potential inside a sphere where the hemispheres are kept at fixed constant potentials +V and -V.



The potential as a series of Legendre polynomials will be

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\vartheta)$$

In addition one has to consider the boundary condition

$$\varphi(R,\vartheta) = \begin{cases} +V & 0 \le \vartheta \le \frac{\pi}{2} \\ -V & \frac{\pi}{2} \le \vartheta \le \pi \end{cases}$$

The coefficients of the series can be written in integral form by using the *orthogonality of Legendre polynomials*

$$\int_{-1}^{1} d\cos\vartheta\, \varphi(R,\vartheta)\, P_m(\cos\vartheta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \underbrace{\int_{-1}^{1} d\cos\vartheta\, P_\ell(\cos\vartheta)\, P_m(\cos\vartheta)}_{\frac{2}{2m+1}\,\delta_{\ell m}}$$

$$\boxed{A_m = \frac{2m+1}{2R^m} \int_{-1}^{1} d\cos\vartheta\, \varphi(R,\vartheta)\, P_m(\cos\vartheta)}$$

$$\hookrightarrow \quad A_m = \frac{2m+1}{2R^m} V \left[\int_0^1 dx\, P_m(x) - \int_{-1}^{0} dx\, P_m(x) \right] = *$$

Observe that

$$P_m(-x) = (-1)^m P_m(x)$$

$$\int_{-1}^{0} dx\, P_m(x) \overset{x' = -x}{=} -\int_{1}^{0} dx'\, P_m(-x') = (-1)^m \int_0^1 dx\, P_m(x)$$

Therefore

$$A_m = \frac{2m+1}{2R^m} V \left[1 - (-1)^m \right] \int_0^1 dx\, P_m(x) \longrightarrow A_m = 0$$

for m even

$$\boxed{A_m = \frac{2m+1}{\cancel{2} R^m} \cancel{2} V \int_0^1 dx\, P_m(x) \qquad \text{for } m \text{ odd}}$$

$$A_1 = \frac{3}{R} V \int_0^1 dx\; x = \frac{3V}{2R}$$

$$A_3 = \frac{7}{R^3} V \int_0^1 dx\, \frac{1}{2}\left(5x^3 - 3x\right) = \frac{7V}{2R^3} \left(\underbrace{\frac{5}{4} - \frac{3}{2}}_{-\frac{1}{4}} \right) = -\frac{7V}{8R^3}$$

etc.

$$\boxed{\varphi(r,\vartheta) = V\left[\frac{r}{R}\, \frac{3}{2}\, P_1(\cos\vartheta) - \frac{r^3}{R^3}\, \frac{7}{8}\, P_3(\cos\vartheta) + \ldots \right]}$$

POTENTIAL INSIDE THE SPHERE

One can also find the potential outside the sphere, by imposing that the potential is zero at infinity. Consequently in the radial part the coefficients A will be zero and the coefficients B will need to be fixed. The considerations concerning the angular integrals will be exactly the same that we wrote for the case of the potential inside the sphere. Therefore one can directly write the solution:

$$\left(\frac{r}{R}\right)^{\ell} \longrightarrow \left(\frac{R}{r}\right)^{\ell+1}$$

$$\boxed{\varphi(r,\vartheta) = V\left[\left(\frac{R}{r}\right)^{2}\frac{3}{2}P_1(\cos\vartheta) - \left(\frac{R}{r}\right)^{4}\frac{7}{8}P_3(\cos\vartheta) + \dots\right]}$$

POTENTIAL OUTSIDE THE SPHERE

Problem 2

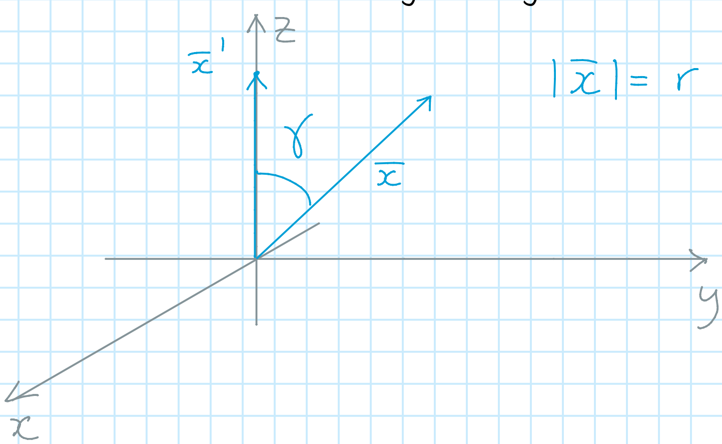
Use the expansion of the potential in Legendre polynomials in order to prove that

$$\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

$$r_< \equiv \min\{|\bar{x}|, |\bar{x}'|\}$$
$$r_> \equiv \max\{|\bar{x}|, |\bar{x}'|\}$$

$$\cos\gamma \equiv \frac{\bar{x}\cdot\bar{x}'}{|\bar{x}||\bar{x}'|}$$

This problem is essentially asking to rewrite the potential of a point particle in a series or Legendre polynomials. Consider the point particle to be placed in $\bar{x}'$, and then imagine to rotate the frame of reference in such a way that $\bar{x}'$ falls on the z axis. The problem has now azimuthal symmetry



$$|\bar{x}| = r$$

Therefore one can write

$$\frac{1}{|\bar{x}-\bar{x}'|} = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\gamma)$$

If also the point $\bar{x}$ is on the z axis one has the additional condition

$$\gamma = 0 \qquad P_\ell\,(\underbrace{\cos\gamma}_{=1}) = 1 \qquad \forall\ \ell$$

$$\left.\frac{1}{|\bar{x}-\bar{x}'|}\right|_{\substack{\bar{x}=r\hat{k}\\ \bar{x}'=r'\hat{k}}} = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) \qquad (a)$$

$$|\bar{x}-\bar{x}'|^2 = \bar{x}^2 + (\bar{x}')^2 - 2\,\bar{x}\cdot\bar{x}' = r^2 + (r')^2 - 2rr'\overbrace{\cos\gamma}^{=1} = (r-r')^2$$

$$\left.\frac{1}{|\bar{x}-\bar{x}'|}\right|_{\substack{\bar{x}=r\hat{k}\\ \bar{x}'=r'\hat{k}}} = \frac{1}{|r-r'|} = \frac{1}{r_>}\ \frac{1}{1-\frac{r_<}{r_>}} = \frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^\ell \qquad (b)$$

Consequently, one can equate (a) and (b)

if $|\bar{x}| = r = r_<$

$$A_\ell r^\ell + B_\ell r^{-(\ell+1)} = \frac{1}{r_>^{\ell+1}}\, r^\ell \longrightarrow \begin{cases} B_\ell = 0 \\ A_\ell = \dfrac{1}{r_>^{\ell+1}} \end{cases}$$

if $|\bar{x}| = r = r_>$

$$A_\ell r^\ell + B_\ell r^{-(\ell+1)} = r_<^\ell\, \frac{1}{r^{\ell+1}} \longrightarrow \begin{cases} A_\ell = 0 \\ B_\ell = r_<^\ell \end{cases}$$

Therefore

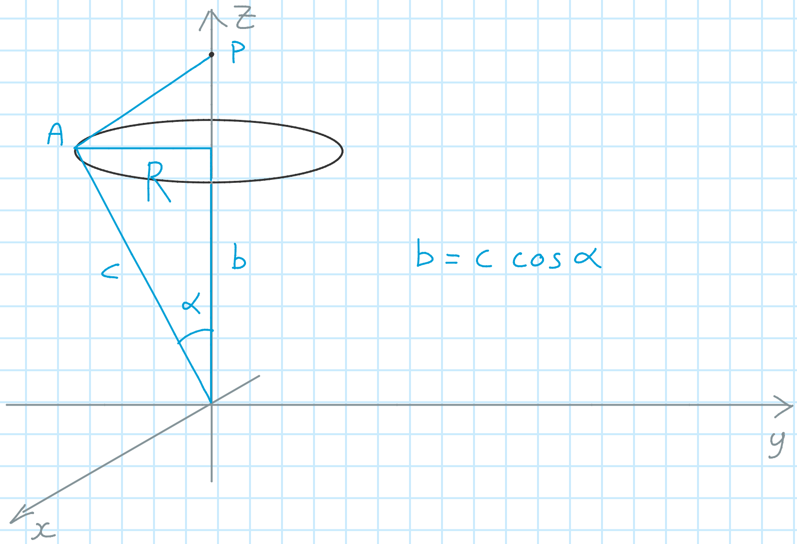
$$\left.\frac{1}{|\bar{x}-\bar{x}'|}\right|_{\substack{\bar{x}=r\hat{k}\\ \bar{x}'=r'\hat{k}}} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}$$

notice that this series diverges if $r_< = r_>$ (and it must diverge)

But the coefficients A and B will not depend on the angle gamma, so that when x is not on the z axis the coefficients A and B have to be the same, and relation we set out to prove at the beginning is valid.

Problem 3

Find the potential due to a total charge q uniformly distributed over a circular ring of radius R. We can choose the reference frame in such a way that the problem has a manifest azimuthal symmetry.



$$b = c\cos\alpha$$

It is possible to find the potential in a generic point in the space surrounding the ring by knowing the potential in a point P along the z axis. The latter depends only on the charge q on the ring and on the distance AP.

$$\overline{AP}^2 = (r - c\cos\alpha)^2 + c^2\sin^2\alpha = r^2 + c^2 - 2cr\sin\alpha$$

$$\varphi(0,0,z\equiv r) = \frac{q}{4\pi\varepsilon_0}\underbrace{\frac{1}{\sqrt{r^2+c^2-2cr\cos\alpha}}}$$

this is a factor $\dfrac{1}{|\bar{x}-\bar{x}'|} = \displaystyle\sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\alpha)$

Notice that alpha is the same angle for all of the points along the ring (azimuthal symmetry). Therefore, for r > c

$$\varphi(0,0,z=r) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{c}{r}\right)^{\ell} P_\ell(\cos\alpha)$$

For $r < c$ instead

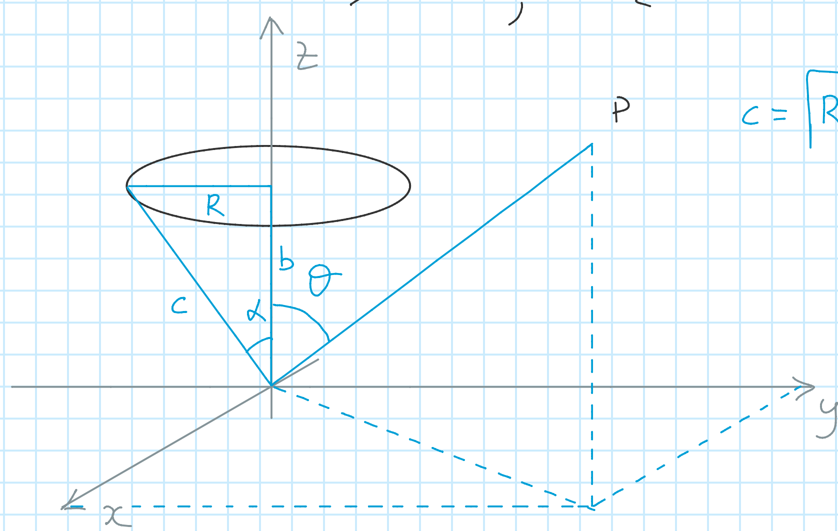
$$\varphi(0,0,z=r) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{c}\sum_{\ell=0}^{\infty}\left(\frac{r}{c}\right)^{\ell} P_\ell(\cos\alpha)$$

The potential at an arbitrary point in space at a distance r from the origin and making an angle theta with the z axis is then obtained by taking the potential for a point on the axis and multiplying each term by the Legendre polynomial of index l and argument theta.

$$\boxed{\varphi(r,\vartheta) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{r_>}\sum_{\ell=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{\ell} P_\ell(\cos\alpha)\, P_\ell(\cos\vartheta)}$$

where, if $r > c$ $\quad r_> = r$, $r_< = c$

if $r < c$ $\quad r_> = c$, $r_< = r$



$c = \sqrt{R^2 + b^2}$

For the special case in which the ring is on the x-y plane

$$\alpha = \frac{\pi}{2} \qquad \cos\alpha = 0 \,, \qquad P_\ell(0) = \begin{cases} 0 & \text{if } \ell = 2m+1 \\ -\dfrac{(2m-1)!!}{2^m\, m!} & \text{if } \ell = 2m \end{cases}$$

rem $\quad n!! = n(n-2)(n-4)\ldots(4)(2)$

$$\boxed{\varphi(r,\vartheta) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{r_>}\sum_{m=0}^{\infty}\left[-\frac{(2m-1)!!}{2^m\,m!}\right]\frac{r_<^{2m}}{r_>^{2m}}\,P_{2m}(\cos\vartheta)}$$