# Multipole expansion - monopole and dipole terms

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The Multipole expansion is typically dominated by the monopole term

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r} \quad , \quad Q_0 = \int d^3r \, \rho(\bar{r})$$

This is exactly the same potential we would find if all of the charge was concentrate in one point. If the total charge is zero, the dominant term is the dipole.

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_1}{r^2} \quad , \quad Q_1 = \int r' \cos\alpha \, \rho(\bar{r}') \, d^3r'$$

Where $\alpha$ is the angle between the vectors r and r'. One can therefore replace r' cos $\alpha$ as follows

$$r' \cos\alpha = \hat{r} \cdot \bar{r}'$$

unit vector in the direction of $\bar{r}$

The dipole potential can then be written as

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat{r} \cdot \underbrace{\int \bar{r}' \rho(\bar{r}') \, d^3r'}_{\bar{p}} = \frac{1}{4\pi\varepsilon_0}\frac{\bar{r} \cdot \bar{p}}{r^3}$$

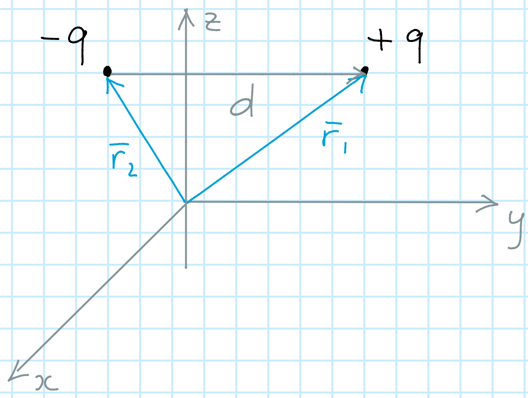
$$\boxed{\bar{p} \equiv \int \bar{r}' \rho(\bar{r}') \, d^3r'}$$

DIPOLE MOMENT

The dipole moment of a collection of point-charges is therefore given by

$$\bar{p} = \sum_{i=1}^{n} q_i \bar{r}_i'$$

From this formula we can reobtain the dipole momentum for a physical dipole



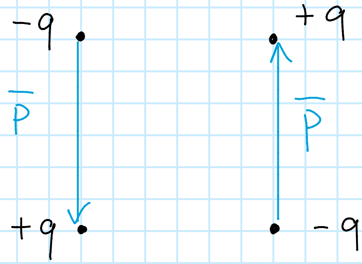

$$\bar{p} = q\bar{r}_1 - q\bar{r}_2 = q(\bar{r}_1 - \bar{r}_2) = q\bar{d}$$

Notice that the potential

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\bar{r}\cdot\bar{p}}{r^3}$$

Is just the approximate potential for a physical dipole, not the exact one. In order to obtain a perfect dipole potential one needs to take the limits d --> 0, q --> ∞ with qd = p constant in the physical dipole case.

Dipoles are vectors and they must be added as vector. For this reason the following charge distribution has zero dipole moment




This distribution is called physical quadrupole.

In the Multipole expansion we can rewrite the quadrupole term by introducing a quadrupole moment (which is a tensor, rather than a vector)

$$\varphi_{quadrupole} = \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r^3}, \quad Q_2 = \int (r')^2 P_2(\cos\alpha)\,\rho(\bar{r}')\,d^3\bar{r}'$$

$$(r')^2 P_2(\cos\alpha) = \frac{r'^2}{2}\left(3\cos^2\alpha - 1\right)$$

$$= \frac{3}{2}(\hat{r}\cdot\bar{r}')^2 - \frac{\bar{r}'\cdot\bar{r}'}{2}$$

$$= \frac{r_i}{r}\frac{r_j}{r}\left[\frac{3}{2} r_i' r_j' - \frac{1}{2}\delta_{ij}(r')^2\right]$$

$$\boxed{Q_{ij} = \frac{1}{2}\int d^3r' \left[3\, r_i' r_j' - \delta_{ij}\,(r')^2\right]\rho(\bar{r}')}$$

QUADRUPOLE MOMENT

$$\varphi_{quadrupole} = \frac{1}{4\pi\varepsilon_0}\frac{r_i r_j}{r^5} Q_{ij}$$

Notice that the quadrupole tensor is symmetric and traceless.
However, things can also be organized in a different way

$$(r')^2 P_2(\cos\alpha) = \frac{(r')^2}{2}\left(3\cos^2\alpha - 1\right) = \frac{(r')^2}{2}\left[3\left(\frac{\hat{r}\cdot\bar{r}'}{r'}\right)^2 - 1\right]$$

$$= \frac{1}{2}\left[3(\hat{r}\cdot\bar{r}')^2 - (r')^2\right] = \frac{1}{2}\left[3\,\hat{r}_i r_i' \hat{r}_j r_j' - r_\ell' r_\ell'\right]$$

$$= \frac{3}{2} r_i' r_j' \left(\hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{ij}\right)$$

$$Q_2 = \int d^3r' \rho(\bar{r}')(r')^2 P_2(\cos\alpha) = \frac{3}{2}\left(\hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{ij}\right) q_{ij}$$

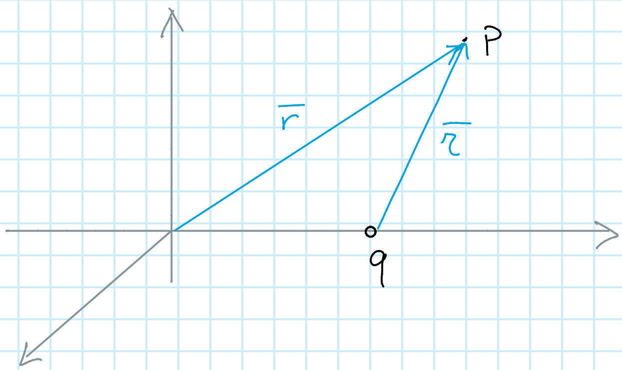
$$\boxed{q_{ij} = \int d^3r' \rho(\bar{r}') r'_i r'_j}$$

"QUADRUPOLE TENSOR"

$$\varphi_{quadrupole} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \frac{3}{2}\left(\hat{r}_i \hat{r}_j - \frac{1}{3}\delta_{ij}\right) q_{ij}$$

## Coordinate system and Multipole expansion

Consider a point like charge which is not placed at the origin of a reference frame




The potential in the point P is not a pure monopole, since we also have the dipole contribution p = q r . Obviously, one could rewrite the potential as a pure monopole by moving the origin of the frame of reference to the position of the charge, so that the potential would become

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{q}{\tau}$$

notice $\tau$, not r !

As expected.

In general the dipole moment (and higher moments) depend on the choice of the frame of reference. The exception to this rule is the case in which the monopole contribution is zero. In fact let's imagine to shift the origin of the frame of reference by a vector a

$$\bar{p} = \int \bar{r}' \rho(\bar{r}')\, d^3r' = \int (\bar{r}'' - \bar{a})\, \overbrace{\rho'(\bar{r}''-a)}^{\equiv \rho(\bar{r}'')}\, d^3r''$$

$$= \underbrace{\int \bar{r}'' \rho(\bar{r}'')\, d^3r''}_{\bar{p} \text{ in the new frame}} - \bar{a} \underbrace{\int \rho(\bar{r}'')\, d^3r''}_{Q_0}$$

So if the monopole is zero, the dipole moment is numerically independent from the choice of the reference frame.

**Example**

Calculate the quadrupole moment tensor with respect to the center of the charge distribution for the spherically symmetric charge distribution

$$\rho(\bar{r}) = k \frac{R}{r^2} (R - 2r) \sin\vartheta \qquad r \leq R$$

In a previous lecture we already showed that this charge distribution has zero monopole and dipole contributions. We actually proved that the dipole contribution is zero for a point along the z axis. However, it is possible to prove that the dipole vector is indeed zero:

$$\bar{p} = \int d^3r\, \bar{r}\, \rho(\bar{r})$$

$$p_x = \int_0^R dr\, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi\, \overbrace{r \sin\vartheta \cos\varphi}^{x}\, k \frac{R}{r^2}(R-2r) \sin\vartheta$$

$$p_x \propto \int_0^{2\pi} d\varphi \cos\varphi = 0$$

similarly $\quad p_y \propto \int_0^{2\pi} d\varphi \sin\varphi = 0$

$$p_z = \int_0^R dr\, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi\, r\cos\vartheta\, k \frac{R}{r^2}(R-2r)\sin\vartheta$$

$$p_z \propto \int_{-1}^{1} d\cos\vartheta \; \cos\vartheta \; \sin\vartheta = \int_{-1}^{1} dx \; x\sqrt{1-x^2} = 0$$

One can then calculate the quadrupole tensor

$$q_{ij} = \underbrace{\int d^3r' \; \rho(\bar{r}') \;\; r_i' r_j'}_{\bar{r}' \text{ dummy variable, drop the prime}} = \int_0^R dr\, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi \; \rho(\bar{r})\, r_i r_j$$

$$r_1 = r\sin\vartheta\cos\varphi \qquad r_2 = r\sin\vartheta\sin\varphi$$

$$r_3 = r\cos\vartheta$$

$$q_{12} = \int_0^R dr\, r^4\, k\frac{R}{r^2}(R-2r)\int_{-1}^{1} d\cos\vartheta \;\; \sin^3\vartheta \; \underbrace{\int_0^{2\pi} d\varphi\, \sin\varphi\, \cos\varphi}_{=0}$$

$$q_{21} = q_{12} = 0$$

$$q_{13} = \int_0^R dr\, r^4\, k\frac{R}{r^2}(R-2r)\int_{-1}^{1} d\cos\vartheta \; \sin^2\vartheta \; \cos\vartheta \; \underbrace{\int_0^{2\pi} d\varphi \; \cos\varphi}_{=0}$$

$$q_{13} = q_{31} = 0$$

$$q_{23} = \int_0^R dr\, r^4\, k\frac{R}{r^2}(R-2r)\int_{-1}^{1} d\cos\vartheta \, \sin^2\vartheta \; \cos\vartheta \quad \underbrace{\int_0^{2\pi} d\varphi\, \sin\varphi}_{=0}$$

$$q_{23} = q_{32} = 0$$

$$q_{11} = \int_0^R dr\; r^{\cancel{4}\,2}\, k\frac{R}{\cancel{r^2}}(R-2r)\int_{-1}^{1} d\cos\vartheta\, \sin^3\vartheta \underbrace{\int_0^{2\pi} d\varphi\, \cos^2\varphi}_{\frac{1}{2}2\pi = \pi}$$

$$\int_{-1}^{1} dx\, (1-x^2)\sqrt{1-x^2} = \frac{3\pi}{8}$$

$$q_{11} = \frac{3\pi^2}{8} \, k R \int_0^R (R - 2r) \, r^2 dr$$

$$= \frac{3\pi^2}{8} k R \left[ R \frac{r^3}{3} - \frac{2}{4} r^4 \right]_0^R = \frac{3\pi^2}{8} k R^5 \left( \frac{1}{3} - \frac{1}{2} \right)$$

$$= \frac{3\pi^2}{8} \, k R^4 \left( -\frac{1}{6} \right) = -\frac{\pi^2}{16} \, k R^5$$

$$q_{11} = q_{22} = -\frac{\pi^2}{16} \, k R^5$$

$$q_{33} = \int_0^R dr \, r^4 \, k \frac{R}{r^2} \, (R - 2r) \int_{-1}^{1} d\cos\vartheta \, \cos^2\vartheta \, \sin\vartheta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\int_{-1}^{1} dx \; x^2 \sqrt{1 - x^2} = \frac{\pi}{8}$$

$$q_{33} = \frac{\pi^2}{4} k R \left( -\frac{R^4}{6} \right) = -\frac{\pi^2}{24} \, k R^5$$

$$Q_2 = \frac{3}{2} \left( \hat{r}_i \hat{r}_j - \frac{1}{3} \delta_{ij} \right) q_{ij}$$

$$= \frac{3}{2} \, k R^5 \Big[ \sin^2\vartheta \, (\cos^2\varphi + \sin^2\varphi) \left( -\frac{\pi^2}{16} \right) + \cos^2\vartheta \left( -\frac{\pi^2}{24} \right)$$

$$+ \frac{1}{3} \frac{2\pi^2}{16} + \frac{1}{3} \frac{\pi^2}{24} \Big]$$

$$= \frac{3}{2} \pi^2 k R^5 \left[ \sin^2\vartheta \left( -\frac{1}{16} \right) + \cos^2\vartheta \left( -\frac{1}{24} \right) + \frac{1}{3} \frac{1}{6} \right]$$

$$= \frac{3}{2} \pi^2 k R^5 \left[ \frac{1}{18} - \frac{1}{24} + \sin^2\vartheta \left( \frac{1}{24} - \frac{1}{16} \right) \right]$$

$$= \frac{3}{2} \pi^2 k R^5 \left[ \frac{1}{72} - \sin^2\vartheta \, \frac{1}{48} \right]$$

$$= \pi^2 k R^5 \left[ \frac{1}{48} - \frac{\sin^2\vartheta}{32} \right]$$

$$\varphi_{quadrupole} = \frac{1}{4\pi\varepsilon_0}\frac{K\pi^2 R^5}{r^3}\left(\frac{1}{48} - \frac{\sin^2\theta}{32}\right)$$

If we look at a point on the z axis (theta = 0) we find the same potential we already found in the problem we solved in a previous lecture.

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One can show in general that the quadrupole tensor for a spherically symmetric charge distribution (calculated with respect to the center of the charge distribution) is proportional to the identity matrix

$$\boxed{\begin{array}{l} \text{if} \quad \rho(\bar{r}) = \rho(r) \\ q_{ij} = 0 \quad \text{for} \quad i \neq j \quad , \quad q_{11} = q_{22} = q_{33} \end{array}}$$

In fact

r, not $\bar{r}$

$$q_{ij} = \int d^3r \; \rho(r) \; r_i r_j$$

$$q_{12} = \int dr \, r^4 \rho(r) \int_{-1}^{1} d\cos\theta \, \sin^2\theta \underbrace{\int_0^{2\pi} d\varphi \, \cos\varphi \sin\varphi}_{=0} = 0$$

$$\int_0^{2\pi} d\varphi \, \cos\varphi = 0 \rightarrow q_{13} = q_{31} = 0$$

$$\int_0^{2\pi} d\varphi \, \sin\varphi = 0 \rightarrow q_{23} = q_{32} = 0$$

The tensor is diagonal, now let's prove that all of the entries are the same

$$q_{33} = \int dr \, r^4 \rho(r) \underbrace{\int_{-1}^{1} d\cos\theta \, \cos^2\theta}_{} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$\int_{-1}^{1} dx \; x^2 = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$$

$$q_{33} = \frac{4\pi}{3} \int dr\, r^4 \rho(r)$$

$$q_{11} = \int dr\, r^4 \rho(r) \underbrace{\int_{-1}^{1} d\cos\vartheta \sin^2\vartheta}_{} \underbrace{\int_0^{2\pi} d\varphi \cos^2\varphi}_{2\pi \frac{1}{2} = \pi}$$

$$\int_{-1}^{1} dx \, (1 - x^2) = x - \frac{x^3}{3} \Big|_{-1}^{1} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$q_{11} = \frac{4\pi}{3} \int dr\, r^4 \rho(r) = q_{33}$$

Finally $q_{33} = q_{22}$ since $\int_0^{2\pi} d\varphi \sin^2\varphi = \int_0^{2\pi} d\varphi \cos^2\varphi = \pi$

$$\boxed{q_{11} = q_{22} = q_{33}}$$

The tensor is diagonal