Dirac delta function

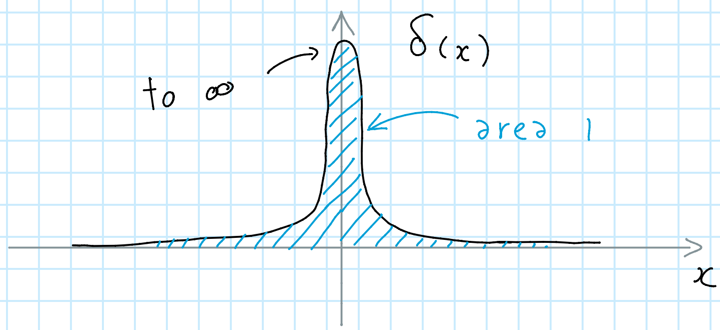
Wednesday, January 30, 2019 9:21 AM

One dimensional Dirac delta function

$$\delta(x) \equiv \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

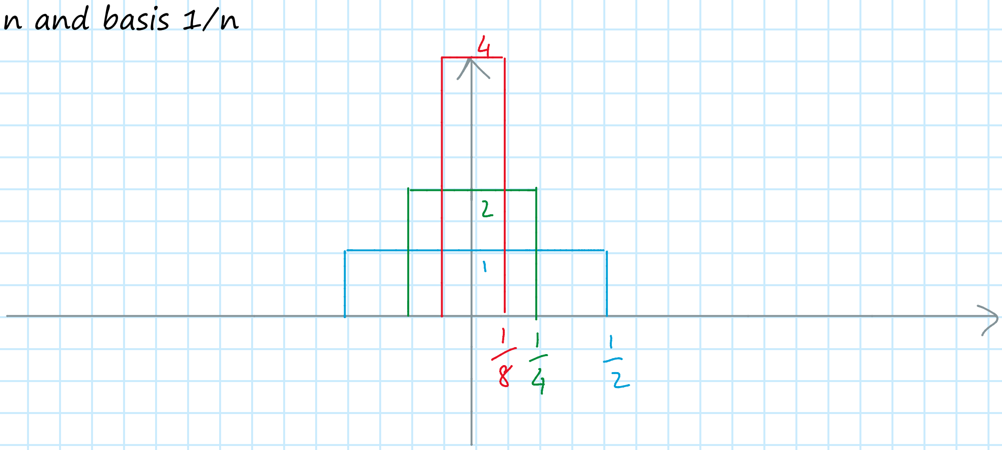
and

$$\int_{-\infty}^{+\infty} dx \; \delta(x) \equiv 1$$

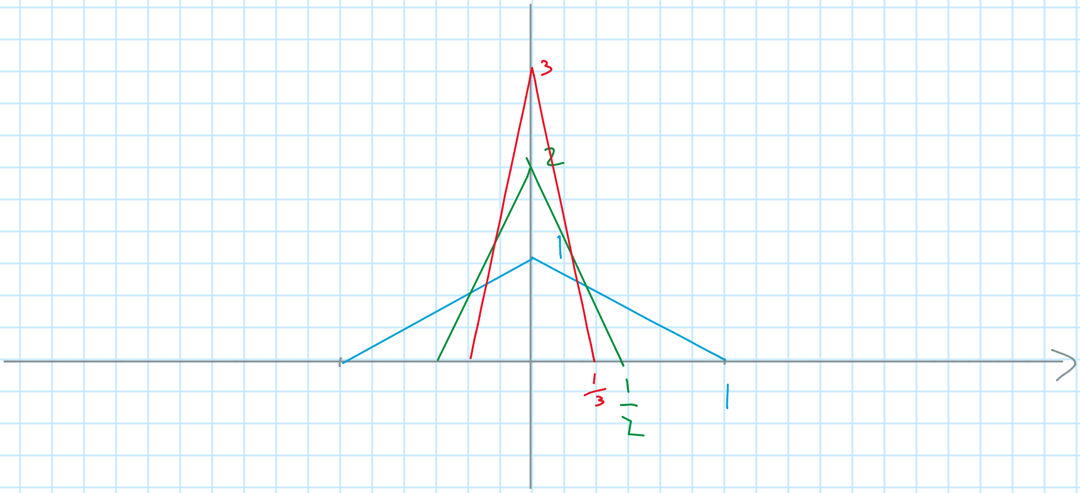


The Delta function is not a function (it is not finite at x = 0); mathematicians refer to it as a generalized function or distribution. It is properly defined as the limit of a series of functions.

Example 1: The delta function is the limit for n--> ∞ of the rectangles R_n, with height n and basis 1/n



Example 2: The delta function is the limit for n --> ∞ of the series of triangles T_n, that are isosceles triangles with height n and base 2/n



Properties

If f(x) is a well behaved function (it is sufficient for it to be continuous) then

$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)$$

The integrand does not need to extend to infinity

$$\int_{-\epsilon}^{+\epsilon} f(x)\,\delta(x)\,dx = f(0)$$

By shifting the argument of the delta function one finds

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

Another important property is that

$$\delta(kx) = \frac{1}{k}\,\delta(x)$$

Proof

$$I = \int_{-\infty}^{+\infty} f(x)\,\delta(kx)\,dx = (*)$$

set $\quad u \equiv kx \qquad dx = \dfrac{du}{k}$

if $\quad k > 0 \quad \begin{cases} x = +\infty \longrightarrow u = +\infty \\ x = -\infty \longrightarrow u = -\infty \end{cases}$

if $\quad k < 0 \quad \begin{cases} x \to +\infty \longrightarrow u = -\infty \\ x \to -\infty \longrightarrow u = +\infty \end{cases}$

$$I = (*) = \frac{1}{k}\int_{\substack{-\infty \\ +\infty}}^{\substack{+\infty \\ -\infty}} du\; f\left(\frac{u}{k}\right)\,\delta(u)$$

The upper sign applies to the case of positive k, the lower sign to the case of negative k. One can then rewrite everything as

$$I = \pm\frac{1}{k}\int_{-\infty}^{+\infty} du\; f\left(\frac{u}{k}\right)\delta(u) = \frac{1}{|k|}\,f(0)$$

This is equivalent to set

$$\delta(kx) = \frac{1}{|k|}\,\delta(x)$$

More in general, one can prove that

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{1}{|f'(x_i)|}\,\delta(x - x_i)$$

for $f(x_i) = 0$

Three dimensional delta function

The three dimensional delta function is defined as

$$\delta^{(3)}(\bar{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

Therefore

$$\int_{\mathbb{R}^3} d^3r \;\; \delta(\bar{r})\, f(\bar{r}) = f(0)$$

$$\int_{\mathbb{R}^3} d^3r \; \delta(\bar{r}-\bar{a})\, f(\bar{r}) = f(\bar{a})$$

Delta function representation

A useful representation of Dirac delta function is

$$\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk \;\; e^{ik(x-x_0)}$$

Should be familiar from Fourier transform

Fourier transform

$$\tilde{f}(k) \equiv \int_{-\infty}^{+\infty} dx \; e^{+ikx}\; f(x)$$

Inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{-ikx} \tilde{f}(k)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{-ikx} \int_{-\infty}^{+\infty} dx' \; e^{+ikx'} f(x')$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dk \; e^{ik(x'-x)} f(x')$$

$$= \int_{-\infty}^{+\infty} dx' \; f(x') \underbrace{\left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{+ik(x'-x)} \right)}_{\delta(x-x')}$$