Poynting Vector

Friday, February 1, 2019 12:44 PM

We already pointed out what is the infinitesimal amount of work done by the fields E and B on a charge q

$$dW = \bar{F} \cdot d\bar{\ell} = \bar{F} \cdot \frac{d\bar{\ell}}{dt} dt = \bar{F} \cdot \bar{v}\, dt$$

$$= q\Big(\bar{E}(\bar{x},t) + \bar{v} \times \bar{B}(\bar{x},t) \Big) \cdot \bar{v}\, dt = q \bar{E} \cdot \bar{v}\, dt$$

The infinitesimal charge can then be written in terms of the charge density and the work in terms of the current density

$$dq = \rho\, d^3x \qquad dW = \rho\, \bar{v} \cdot \bar{E}\, d^3x\, dt = \bar{J} \cdot \bar{E}\, d^3x\, dt$$

$$\boxed{\bar{J} \cdot \bar{E} \equiv \ell}$$

POWER DENSITY

$$[\ell] = \underbrace{\frac{C}{m^3}}_{\rho} \underbrace{\frac{\cancel{m}}{s}}_{\bar{v}} \underbrace{\frac{V}{\cancel{m}}}_{\bar{E}} = \frac{kg \frac{m^2}{s^2}}{m^3\, s} = \frac{Watt}{m^3}$$

One can now try to rewrite the power density as a function of the fields. In order to achieve this goal, one can start from the Ampere Maxwell's law

$$\nabla \times \bar{B} - \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} = \mu_0 \bar{J}$$

$$\ell = \bar{J} \cdot \bar{E} = \frac{1}{\mu_0} \left(\nabla \times \bar{B} - \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} \right) \cdot \bar{E}$$

One can now use the relation

$$\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot (\nabla \times \bar{a}) - \bar{a} \cdot (\nabla \times \bar{b})$$

$$\nabla \cdot (\bar{E} \times \bar{B}) = \bar{B} \cdot (\underbrace{\nabla \times \bar{E}}_{= -\frac{\partial \bar{B}}{\partial t}}) - \bar{E} \cdot (\nabla \times \bar{B})$$

Faraday's law

$$\bar{E} \cdot (\nabla \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{E}) - \nabla \cdot (\bar{E} \times \bar{B})$$

$$= -\bar{B} \cdot \frac{\partial \bar{B}}{\partial t} - \nabla \cdot (\bar{E} \times \bar{B})$$

Consequently

$$\ell = \frac{1}{\mu_0} \left[-\bar{B} \cdot \frac{\partial \bar{B}}{\partial t} - \nabla \cdot (\bar{E} \times \bar{B}) - \frac{1}{c^2} \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} \right]$$

$$= -\frac{1}{\mu_0} \bar{B} \cdot \frac{\partial \bar{B}}{\partial t} - \frac{1}{\mu_0} \nabla \cdot (\bar{E} \times \bar{B}) - \varepsilon_0 \bar{E} \cdot \frac{\partial \bar{E}}{\partial t}$$

It is now convenient to introduce the energy density per unit volume

$$\boxed{w \equiv \frac{1}{2} \varepsilon_0 \bar{E}^2 + \frac{1}{2\mu_0} \bar{B}^2}$$

$$[\varepsilon_0 E^2] = \frac{C}{\cancel{V} m} \frac{V^{\cancel{2}}}{m^2} = \frac{kg \frac{m^2}{s^2}}{m^3}$$

✓

$$[B] = \frac{N}{C\,\frac{m}{s}} = \frac{kg\,\frac{m}{s^2}}{A\,m} = \frac{kg}{A\,s^2}$$

$$[\mu_0] = \frac{Vs}{Am} = \frac{C\,V\,s}{C\,A\,m} = kg\,\frac{m^{\cancel{2}}}{s^2}\,\frac{s}{C}\,\frac{1}{A\,\cancel{m}} = \frac{kg\,m}{A^2\,s^2}$$

$$\left[\frac{1}{\mu_0}B^2\right] = \underbrace{\frac{\cancel{s^2}\,\cancel{A^2}}{\cancel{kg}\,m}}_{\frac{1}{\mu_0}}\;\underbrace{\frac{kg^{\cancel{2}}}{s^{\cancel{4}\,2}\,\cancel{A^2}}}_{B} = \frac{kg\,\frac{m^2}{s^2}}{m^3} \qquad \checkmark$$

In addition, it is useful to define the vector

$$\boxed{\bar{S} = \frac{1}{\mu_0}\,\bar{E}\times\bar{B}}$$

POYNTING VECTOR

(JOHN HENRY POYNTING 1852 - 1914)

$$[S] = \frac{\cancel{s^2}\,A^{\cancel{2}}}{\cancel{kg}\,m}\;\frac{V}{m}\;\frac{\cancel{kg}}{\cancel{s^2}\,\cancel{A}} = \frac{C}{s}\,\frac{V}{m^2} = \frac{kg\,\frac{m^2}{s^2}}{s\,m^2} = \frac{Watt}{m^2}$$

$$= \frac{energy}{time\ area} = \frac{power}{area}$$

With the definitions above one finds that

$$\ell = \bar{J}\cdot\bar{E} = -\frac{\partial}{\partial t}w - \nabla\cdot\bar{S}$$

Consequently,

$$\boxed{\frac{\partial w}{\partial t} + \nabla\cdot\bar{S} + \bar{J}\cdot\bar{E} = 0}$$

POYNTING LAW

$$\boxed{\text{Case in which } j = 0}$$

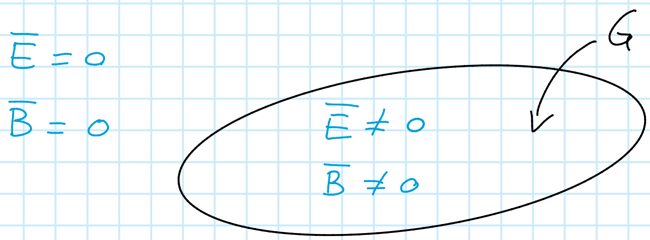
If the charge density vanishes, the Poynting law becomes

$$\frac{\partial w}{\partial t} + \nabla \cdot \bar{S} = 0$$

The equation above is similar in form to the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{j} = 0$$

In addition, let's assume that the fields are non zero only in a large volume G, while they vanish outside G



One can now define the total energy at the time t as

$$W(t) = \int_{\mathbb{R}^3} d^3x \quad w(\bar{x}, t)$$

$$\frac{d}{dt} W = \int_{\mathbb{R}^3} d^3x \ \frac{\partial w}{\partial t} = - \int_{\mathbb{R}^3} d^3x \ \nabla \cdot \bar{S} = - \int_G d^3x \ \nabla \cdot \bar{S}$$

$$= \oint_{\partial G} d\bar{s} \cdot \bar{S} = 0$$

Gauss' theorem

∂G is the surface of G, where $\bar{E}$ and $\bar{B}$ vanish

W is time independent.

Case in which $j \neq 0$

$$\frac{d}{dt} W = -\int_G d^3x \; \nabla \cdot \overline{S} - \int_G d^3x \; \overline{j} \cdot \overline{E} = -\int_G d^3x \; \overline{j} \cdot \overline{E}$$

Since E is zero outside G, one can also write

$$\frac{dW}{dt} = -\int_{\mathbb{R}^3} d^3x \quad \overline{j} \cdot \overline{E}$$

ENERGY TRANSFERRED FROM FIELDS TO CHARGES PER UNIT OF TIME

Observations

1) W does not vanish unless both fields B and E vanish
2) The Poynting vector does not vanish unless

$$\overline{E} = 0 \quad \text{or} \quad \overline{B} = 0 \quad \text{or} \quad \overline{E} \,//\, \overline{B}$$

3) E,B and S form a right handed set of axes