Potential and gauge transformations

Wednesday, August 29, 2018 4:46 PM

The homogeneous ME do not depend on the sources ρ and $\bar{J}$ can be solved in general
Since $\nabla\cdot(\nabla\times\bar{A})=0$ always, one can introduce a vector potential such that

$$\boxed{\bar{B}(\bar{x},t)\equiv\nabla\times\bar{A}(\bar{x},t)}$$

Now plug the relation above in the other homogeneous equation

$$\nabla\times\bar{E}+\frac{\partial\bar{B}}{\partial t}=0$$

$$\hookrightarrow\quad \nabla\times\left(\bar{E}+\frac{\partial\bar{A}}{\partial t}\right)=0$$

The field $\bar{E}+\frac{\partial\bar{A}}{\partial t}$ has zero curl, therefore it can be written as a gradient

$$\bar{E}+\frac{\partial\bar{A}}{\partial t}\equiv-\nabla\varphi\quad;\quad\varphi(\bar{x},t)\rightarrow\text{SCALAR POTENTIAL}$$

$$\boxed{\bar{E}=-\nabla\varphi-\frac{\partial\bar{A}}{\partial t}}$$

Observe: Several different scalar and vector potentials lead to the same fields E and B, which are the physical quantities

Theorem

If two different potential pairs $(\varphi,\bar{A})$ and $(\varphi',\bar{A}')$ lead to the same fields E and B, then the scalar function $f(\bar{x},t)$ exists, such as that

$$\varphi'=\varphi-\frac{\partial f}{\partial t}\qquad\qquad\bar{A}'=\bar{A}+\nabla f$$

Proof

let's define $\bar{A}_1=\bar{A}'-\bar{A}$ and $\varphi_1=\varphi'-\varphi$
then

$$\nabla\times\bar{A}_1=\underbrace{\nabla\times\bar{A}'}_{\bar{B}}-\underbrace{\nabla\times\bar{A}}_{\bar{B}}=0$$

$$\nabla\varphi_1 + \frac{\partial \bar{A}_1}{\partial t} = \Big(\underbrace{\nabla\varphi' + \frac{\partial \bar{A}'}{\partial t}}_{-\bar{E}}\Big) - \Big(\underbrace{\nabla\varphi + \frac{\partial \bar{A}}{\partial t}}_{-\bar{E}}\Big) = 0$$

Consequently, φ_1 and $\bar{A}_1$ are potentials for $\bar{E}=0$, $\bar{B}=0$

Therefore

$$\nabla\times\bar{A}_1 = 0 \longrightarrow \bar{A}_1 = \nabla h$$

and

$$\nabla\varphi_1 + \frac{\partial}{\partial t}\bar{A}_1 = \nabla\varphi_1 + \nabla\frac{\partial}{\partial t}h = 0$$

$$\hookrightarrow \quad \nabla\Big(\underbrace{\varphi_1 + \frac{\partial h}{\partial t}}_{\equiv g}\Big) = 0$$

The function g therefore does not depend on the position but only on time. Now let's define another function f as

$$f = h - \int_0^t g(t')\,dt'$$

$$\nabla f = \nabla h = \bar{A}_1 = \bar{A}' - \bar{A} \qquad (a)$$

$$\frac{\partial f}{\partial t} = \frac{\partial h}{\partial t} - g = \frac{\partial h}{\partial t} - \Big(\varphi_1 + \frac{\partial h}{\partial t}\Big) = -\varphi_1 = -(\varphi' - \varphi) \qquad (b)$$

From (a) and (b) one finds

$$\bar{A}' = \bar{A} + \nabla f \quad ; \quad \varphi' = \varphi - \frac{\partial f}{\partial t}$$

Q.E.D.

$$\boxed{\begin{aligned} \varphi &\rightarrow \varphi' = \varphi - \frac{\partial f}{\partial t} \\ \bar{A} &\rightarrow \bar{A}' = \bar{A} + \nabla f \end{aligned}}$$

Gauge transformations

E and B are invariant under gauge transformations; electrodynamics is a gauge theory. Also the theories describing weak and strong interactions are gauge theories.

Gauge choices

One can choose to work in a particular gauge:

$$\nabla\cdot\bar{A}' = 0 \quad \longrightarrow \quad \Delta f = -\nabla\cdot\bar{A} \qquad \text{RADIATION GAUGE}$$

When studying the propagation of EM waves it is often convenient to work in a gauge where

$$\nabla\cdot\bar{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \qquad \text{LORENTZ GAUGE}$$

Problem: Show that in the Lorentz gauge the scalar and vector potentials satisfy wave equations of the form

$$\square\varphi = \frac{1}{\varepsilon_0}\rho \quad ; \quad \square\bar{A} = \mu_0\bar{J}$$

Where the D' Alambert operator is defined as

$$\square \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$$

Solution:

$$\nabla\cdot\bar{E} = \frac{\rho}{\varepsilon_0} \longrightarrow -\nabla\cdot(\nabla\varphi) - \frac{\partial}{\partial t}\nabla\cdot\bar{A} = \frac{\rho}{\varepsilon_0}$$

but in Lorentz gauge $\nabla\cdot\bar{A} = -\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$

therefore

$$-\Delta\varphi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi = \frac{\rho}{\varepsilon_0} \longrightarrow \square\varphi = \frac{\rho}{\varepsilon_0}$$

Similarly

$$\nabla \times \bar{B} - \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t} = \mu_0 \bar{J} \longrightarrow \nabla \times (\nabla \times \bar{A}) - \frac{1}{c^2}\left(-\frac{\partial}{\partial t}\nabla\varphi - \frac{\partial^2 \bar{A}}{\partial t^2}\right) = \mu_0 \bar{J}$$

remember: $\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \Delta \bar{A}$, so that

$$\nabla(\nabla \cdot \bar{A}) - \Delta \bar{A} + \frac{1}{c^2}\nabla \frac{\partial}{\partial t}\varphi + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{A} = \mu_0 \bar{J}$$

$$\nabla \Big(\underbrace{\nabla \cdot \bar{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\varphi}_{=0 \text{ in LORENTZ GAUGE}}\Big) + \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} - \Delta \bar{A} = \mu_0 \bar{J}$$

$$\Box \bar{A} = \mu_0 \bar{J}$$

Q.E.D.