Separation of variables - cartesian 2D

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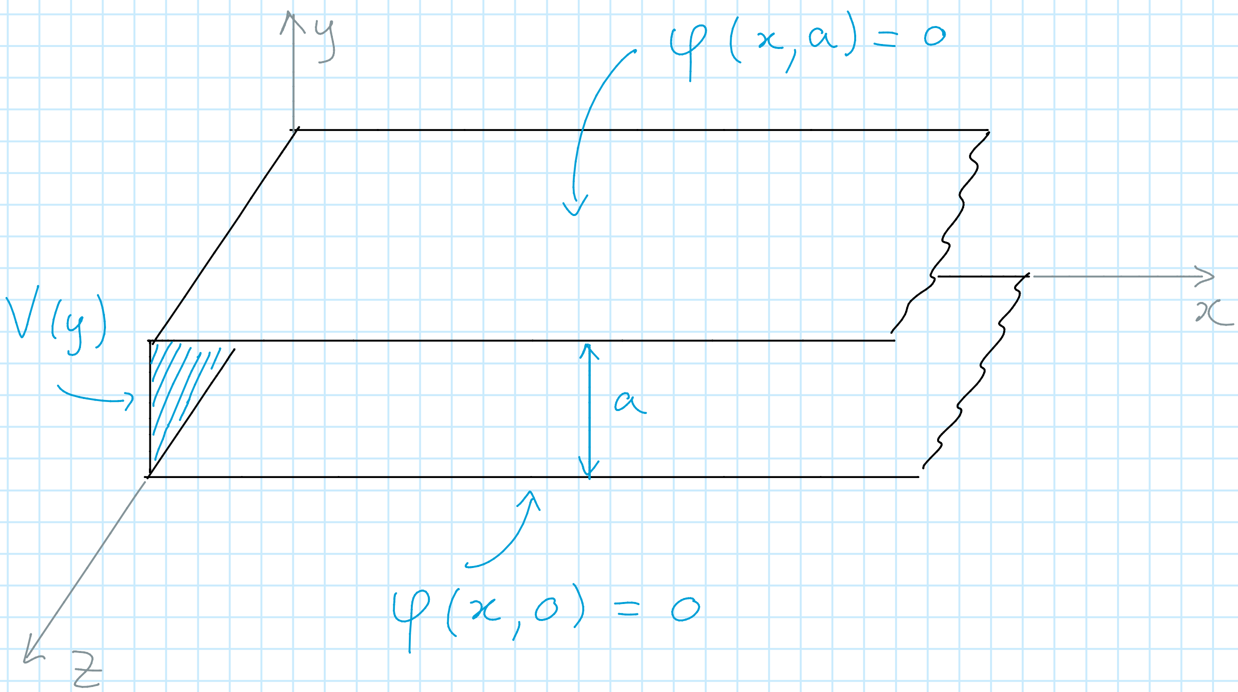
A powerful method for the solution of partial differential equations (such as Laplace equation) is given by the method of separation of variables. Following Griffith (section 3), we start by analyzing a problem which is effectively two dimensional. We consider the region between two grounded plates separate by a distance a. The space is closed off at x = 0, by a strip where the potential is given by some function V(y). The variable z does not play any role in this problem



Since the plates are infinitely long in the z direction, the problem has a symmetry with respect to translations along the z axis, and the potential in the slot between the plates cannot depend on z. Consequently, the equation which one needs to solve is a two dimensional Laplace equation

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

With the boundary conditions

some given function

$$\varphi(x,0) = 0 \quad , \quad \varphi(x,a) = 0 \, , \quad \varphi(0,y) = V(y)$$

In addition, we might expect the potential to go to vanish for x --> ∞, infinitely far away from the strip at x = 0. Therefore we add the request that

$$\varphi(\infty, y) = 0$$

We now look for a special set of solutions for the Laplace equation, a solution where the x and y dependence factorizes in the product of two functions:

$$\varphi(x, y) = X(x)\, Y(y)$$

We will see that we can build a generic solution for the Laplace equation by stitching together special solutions of the factorized form indicated above.
We now try to find out what is the form of the factorized solution. The Laplace equation becomes

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

By dividing everything by XY one finds

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\underbrace{\frac{1}{X} \frac{\partial^2 X}{\partial x^2}}_{\text{can depend only on } x} = - \underbrace{\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}}_{\text{can depend only on } y}$$

the equality should be valid for $\forall x, y$

$$\Rightarrow \quad \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = C \qquad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -C$$

Where C is some constant. For reasons that will become clear shortly, let's choose C = k^2, where k is another constant.

$$\boxed{\frac{\partial^2 X}{\partial x^2} = k^2 X} \qquad \boxed{\frac{\partial^2 Y}{\partial y^2} = -k^2 Y}$$

If we assume k^2 to be positive, the solution for X is an exponential, while the solution for Y is a sinusoidal

$$X = A e^{kx} + B e^{-kx}$$

$$Y = C \sin ky + D \cos ky$$

Therefore the generic factorized solution of the Laplace equation is

$$\varphi(x,y) = \left(A e^{kx} + B e^{-kx}\right)\left(C \sin ky + D \cos ky\right)$$

Now we need to impose the boundary conditions. Since the potential needs to vanish at x --> ∞, one must have A = 0. Furthermore, since the potential is 0 at y = 0 for every x, one must have D = 0. The solution at this stage looks like

$$\varphi(x,y) = C e^{-kx} \sin ky$$

B was absorbed in C. In order to have a vanishing potential at y = a one needs

$$\sin ka = 0 \longrightarrow k = \frac{n\pi}{a} \qquad n = 1, 2, 3, \dots$$

Remember that we need a positive k in order to have a decreasing exponential, therefore n > 0. n=0, which implies k = 0 does not work because in that case the potential would be zero everywhere.

So far we found a family of solutions of the Laplace equations which satisfy 3 of the 4 boundary conditions which we have. Now we observe that since the Laplace equation is linear, any linear combination of these special solutions satisfies the

Laplace equation. Therefore we can work with a solution of the form

$$\varphi(x,y) = \sum_{n=1}^{\infty} C_n \, e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$$

In addition, we impose the last boundary condition and require that

$$\varphi(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} y\right) = V(y)$$

The equation above is a Fourier series, and we know that any physically reasonable potential V can be written as a Fourier series.

We now want a method to find the coefficients C when V is given. In order to find the Cs we apply the following trick

$$\int_0^a \sin\left(\frac{m\pi}{a} y\right) \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} y\right) dy = \int_0^a V_0(y) \sin\left(\frac{m\pi}{a} y\right) dy$$

The sin functions are orthogonal on the interval [0,a]

$$\int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy = \begin{cases} 0 & \text{if } m \neq n \\ \frac{a}{2} & \text{if } m = n \end{cases}$$

Therefore

$$\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy}_{\frac{a}{2}\delta_{mn}} = \int_0^a V(y) \sin\left(\frac{m\pi}{a} y\right) dy$$

$$\boxed{C_m = \frac{2}{a} \int_0^a V(y) \sin\left(\frac{m\pi}{a} y\right) dy}$$

Let's now consider the simple case in which the function V is simply a constant. In that case

$$C_n = \frac{2V}{a}\int_0^a \sin\left(\frac{n\pi}{a}y\right)dy$$

$$= \frac{2V}{a}\,\frac{a}{n\pi}\int_0^{n\pi}\sin u\,du$$

$$= \frac{2V}{n\pi}\Big[-\cos u\Big]_0^{n\pi} = \frac{2V}{n\pi}\left[-(-1)^n+1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{4V}{n\pi} & n \text{ odd} \end{cases}$$

So that finally

$$\varphi(x,y) = \frac{4V}{\pi}\sum_{n\ \mathrm{odd}}\frac{1}{n}e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

(set $x=0$ and plot the first few terms)

The series can indeed be summed exactly (we just quote the result)

$$\varphi(x,y) = \frac{2V}{\pi}\arctan\left(\frac{\sin\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)}\right)$$

(plot this with Mathematica)

at $y=0$, $x \neq 0$ the equation above gives

$$\varphi(x,0) = \frac{2V}{\pi} \arctan(0) = 0 \quad \checkmark$$

same at $y=a$

At $x=0$, $y \neq 0$

$$\varphi(0,y) = \frac{2V}{\pi} \arctan\left(\frac{\sin\left(\frac{\pi y}{a}\right)}{0}\right) = V \quad \checkmark$$