Legendre Polynomials

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Legendre polynomials are solutions of the differential equation

$$\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{dP}{d\vartheta}\right) + \ell(\ell+1)\sin\vartheta\, P = 0$$

Where $l = 0,1,2,\ldots$

Legendre polynomials are the only solutions of the differential equation above which are regular at $\vartheta = 0$, $\vartheta = \pi$

Legendre polynomials are labeled by the subscript l and can be generated through the Rodriguez formula

$x \equiv \cos\vartheta$

$$\boxed{P_\ell(x) = \frac{1}{2^\ell\,\ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell}$$

The first five polynomials are

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2-1)$$
$$P_3(x) = \frac{1}{2}(5x^3-3x)$$
$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$
$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$$

The label l indicates the order of the polynomial. When l is even the polynomial includes only even powers of l, when l is odd, the polynomial includes only odd powers of l.

The numerical prefactor in Rodriguez formula is chosen in such a way that

$$P_\ell(1) = 1 \qquad \forall\, \ell$$

Legendre polynomials are a complete set of functions in the interval $-1 < x < 1$
In addition, the polynomials are orthogonal

$$\int_{-1}^{1} P_\ell(x)\, P_m(x)\, dx = \int_0^\pi P_\ell(\cos\vartheta)\, P_m(\cos\vartheta)\, \sin\vartheta\, d\vartheta$$

$$= \begin{cases} 0 & \text{if } \ell \neq m \\ \dfrac{2}{2\ell+1} & \text{if } \ell = m \end{cases}$$

For our purposes it is important to remember that Legendre polynomials appear in the Taylor expansion of the inverse of the distance between two points. Jackson often uses the notation

$$\frac{1}{|\bar{x} - \bar{y}|} = \frac{1}{r_>}\left[1 - \left(\frac{r_<}{r_>}\right)^2 - \frac{r_<}{r_>}\cos\gamma\right]^{-\frac{1}{2}}$$

$$\boxed{\frac{1}{|\bar{x} - \bar{y}|} = \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\gamma)}$$

with $\quad r_< = \min\{|\bar{x}|, |\bar{y}|\}, \quad r_> = \max\{|\bar{x}|, |\bar{y}|\}$

$$\bar{x} \cdot \bar{y} = r_< r_> \cos\gamma$$

The inverse of the distance between the two points is called the generating function of the Legendre polynomials.