Electric Dipole

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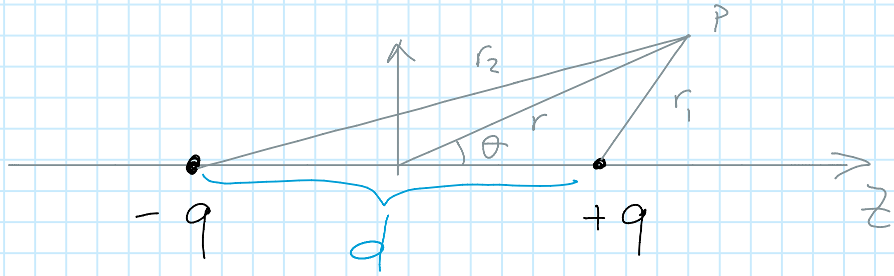
A particularly interesting combination of charges is the electric dipole. A dipole is the name one gives to two charges of equal magnitude but of opposite sign placed at a distance d from one another



One can easily calculate the potential due to this charge distribution

$$r_1^2 = \left(z - \frac{d}{2}\right)^2 + x^2 + y^2 = z^2 + x^2 + y^2 - zd + \frac{d^2}{4} = r^2 - zd + \frac{d^2}{4}$$

$$r_2^2 = \left(z + \frac{d}{2}\right)^2 + x^2 + y^2 = r^2 + zd + \frac{d^2}{4}$$

$$\varphi = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 - zd + \frac{d^2}{4}}} - \frac{1}{\sqrt{r^2 + zd + \frac{d^2}{4}}}\right) = *$$

$$r^2 - zd + \frac{d^2}{4} = r^2 - r\cos\vartheta\, d + \frac{d^2}{4} = r^2\left(1 - \frac{d\cos\vartheta}{r} + \frac{d^2}{4r^2}\right)$$

$$\frac{1}{\sqrt{r^2 - zd + \frac{d^2}{4}}} = \frac{1}{r}\left[1 - \underbrace{\left(\frac{d\cos\vartheta}{r} - \frac{d^2}{4r^2}\right)}_{\varepsilon}\right]^{-\frac{1}{2}}$$

$$\frac{1}{\sqrt{r^2 + zd + \frac{d^2}{4}}} = \frac{1}{r}\left[1 + \underbrace{\left(\frac{d\cos\vartheta}{r} + \frac{d^2}{4r^2}\right)}_{\delta}\right]^{-\frac{1}{2}}$$

Assume now $r \gg d$

remember

$$\frac{1}{\sqrt{1-\varepsilon}} = 1 + \frac{\varepsilon}{2} + \frac{3}{8}\varepsilon^2 + \dots = \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \varepsilon^n$$

$$\binom{-\frac{1}{2}}{n} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma(n+1)\,\Gamma\left(n+\frac{1}{2}+1\right)}$$

$$\frac{1}{\sqrt{1+\delta}} = 1 - \frac{\delta}{2} + \frac{3}{8}\delta^2 + \dots = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \delta^n$$

$$\frac{1}{\sqrt{r^2 - zd + \frac{d^2}{4}}} - \frac{1}{\sqrt{r^2 + zd + \frac{d^2}{4}}} =$$

$$= \frac{1}{r}\left(1 + \frac{1}{2}\left(\frac{d\cos\vartheta}{r} - \frac{d^2}{4r^2}\right) - 1 + \frac{1}{2}\left(\frac{d\cos\vartheta}{r} + \frac{d^2}{4r^2}\right) + \dots\right)$$

$$= \frac{d\cos\vartheta}{r^2} + \dots$$

Therefore

$$\varphi = (*) = \frac{1}{4\pi\varepsilon_0} \frac{qd\cos\vartheta}{r^2} + \dots$$

The terms indicated with the dots vanish if we take the limit d --> 0. However, this would send to zero also the first term. In order to keep the first term we need to consider two simultaneous limits, d -> 0 and q -> Infinity, in such a way that qd is a constant. This is what is usually referred to as dipole potential. In addition, we can call p = qd a dipole moment and write the dipole potential as

$$\varphi = \frac{1}{4\pi\varepsilon_0} \frac{p\cos\vartheta}{r^2}$$

The dipole moment is indeed a vector of magnitude p directed from the negative charge to the positive charge.

It is now of interest to evaluate the electric field due to the dipole potential. Let's assume that the dipole is oriented along the z axis and let's work in spherical coordinates.

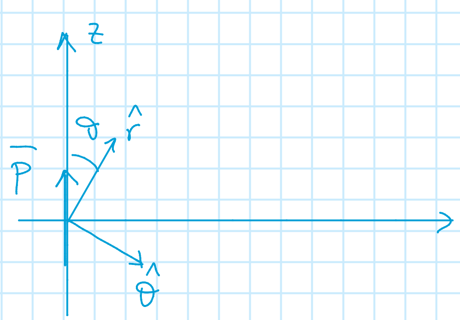
$$\bar{p} = p\,\hat{k}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\hat{\vartheta} + \frac{1}{r}\frac{1}{\sin\vartheta}\frac{\partial f}{\partial \varphi}\hat{\varphi}$$

$$\bar{E} = -\nabla\varphi = -\nabla\left(\frac{1}{4\pi\varepsilon_0}\frac{p\cos\vartheta}{r^2}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\left(2\frac{p\cos\vartheta}{r^3}\hat{r} + \frac{p}{r^3}\sin\vartheta\,\hat{\vartheta}\right)$$

Then observe that



$$\bar{p} = p\cos\vartheta\,\hat{r} - p\cos\left(\frac{\pi}{2} - \vartheta\right)\hat{\vartheta}$$

$$= p\cos\vartheta\,\hat{r} - p\sin\vartheta\,\hat{\vartheta}$$

Therefore

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left(2p\cos\vartheta\,\hat{r} + p\sin\vartheta\,\hat{\vartheta} + p\cos\vartheta\,\hat{r} - p\cos\vartheta\,\hat{r}\right)$$

$$\boxed{\bar{E} = \frac{1}{4\pi\varepsilon_0}\frac{(3\,\bar{p}\cdot\hat{r})\,\hat{r} - \bar{p}}{r^3}}$$

DIPOLE
ELECTRIC
FIELD

Further reading: Force between two dipoles