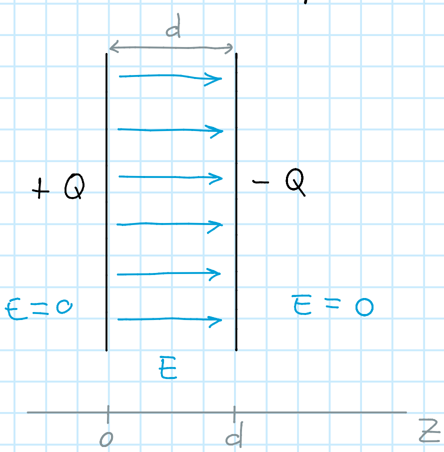
Capacitors

Sunday, November 18, 2018 5:23 PM

Capacitors are systems formed by two conductors, one carrying charge +Q, the other carrying charge -Q. They often appear in technological applications and in problems, since they can use to store energy. We consider below two simple geometric arrangements.

Parallel plate capacitor

A parallel plate capacitor is made of two flat parallel surfaces carrying opposite charges. One can consider the case in which the distance d between the plates is much smaller than the sides of the plates, and look at the electric field between the plates far away from the edges ignoring effects which become relevant only at the boundaries. By using what was discussed in relation to the electric field due to a uniform charge distribution on a flat surface one can conclude that the field between the plates is



$$\overline{E} = \frac{\sigma}{\varepsilon_0}\hat{k}$$

$$\sigma = \frac{Q}{A}$$

A = area of the plates

The capacitance C is defined as (V is the potential difference between the plates)

$$C = \frac{Q}{V}$$

units → Farad (F)

$$F = \frac{C}{m^2}$$

The voltage at a point z between the plates is

$$\varphi = -Ez + \text{const} \qquad V = \varphi(0) - \varphi(d) = Ed = \frac{Qd}{A\varepsilon_0}$$

Therefore the capacitance depends only on the geometric properties of the capacitor

$$C = \frac{Q}{V} = \frac{A\varepsilon_0}{d}$$

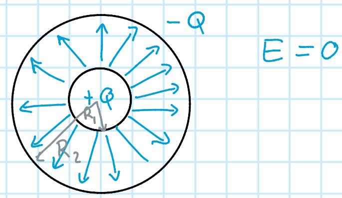
The energy stored in the capacitor is

$$U = \frac{\varepsilon_0}{2}\int d^3r \; \bar{E}\cdot\bar{E} = \frac{A\varepsilon_0}{2}\int_0^d dz \; \frac{\sigma^2}{\varepsilon_0^2} = \frac{A\sigma^2 d}{2\varepsilon_0}$$

$$= \frac{A^2\sigma^2}{2}\;\frac{d}{A\varepsilon_0} = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

Spherical capacitor

Consider the case of two conducting spheres with the same center



By simply applying Gauss' theorem one finds that the field between the spheres is

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\;\frac{Q}{r^2}\;\hat{r} \qquad\qquad R_1 < r < R_2$$

And the potential difference is

$$-V = \varphi(R_2) - \varphi(R_1) = -\int \bar{E}\cdot d\bar{\ell} = -\int_{R_1}^{R_2} dr \; \frac{1}{4\pi\varepsilon_0}\;\frac{Q}{r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0}\;\frac{R_2 - R_1}{R_1 R_2}$$

Consequently, the capacitance is

$$C = \frac{Q}{V} = 4\pi\varepsilon_0\;\frac{R_1 R_2}{R_2 - R_1}$$

Again, we verified a posteriori that the capacitance depends only on the capacitor's geometry.