Integral Vector Calculus

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Fundamental theorem of calculus

It relates indefinite and definite integrals. If

$$\int f(x)\,dx = F(x)$$

Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In a more compact way, one can restate the theorem as

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

Fundamental theorem for gradients

(no proof here)

$$\bar{a} \equiv \{x_a, y_a, z_a\} \qquad \bar{b} = \{x_b, y_b, z_b\}$$

$$\boxed{\int_{\bar{a}}^{\bar{b}} (\nabla f) \cdot d\bar{\ell} = f(\bar{b}) - f(\bar{a})}$$

Corollary 1

$$\int_{\bar{a}}^{\bar{b}} (\nabla f) \cdot d\bar{\ell}$$ is independent from the path taken from $\bar{a}$ to $\bar{b}$

Corollary 2

$$\oint (\nabla f) \cdot d\bar{\ell} = 0$$ since in a closed path the initial and final points coincide

Gauss theorem

(no proof here)

(can be thought of as the fundamental theorem for divergences)

$$\int_{V} \nabla \cdot \bar{v} \, d^3x = \oint_{\partial V} \bar{v} \cdot d\bar{s}$$

closed volume

surface of the volume V

The integral of a divergence of a vector over a closed volume is equal to the flux of the vector over the surface that bounds the volume.

Geometrical interpretation

$$\int \text{sources within } V = \oint \text{flow through the surface of } V$$

Stoke's theorem

(no proof here)

(can be interpreted as the fundamental theorem for curls)

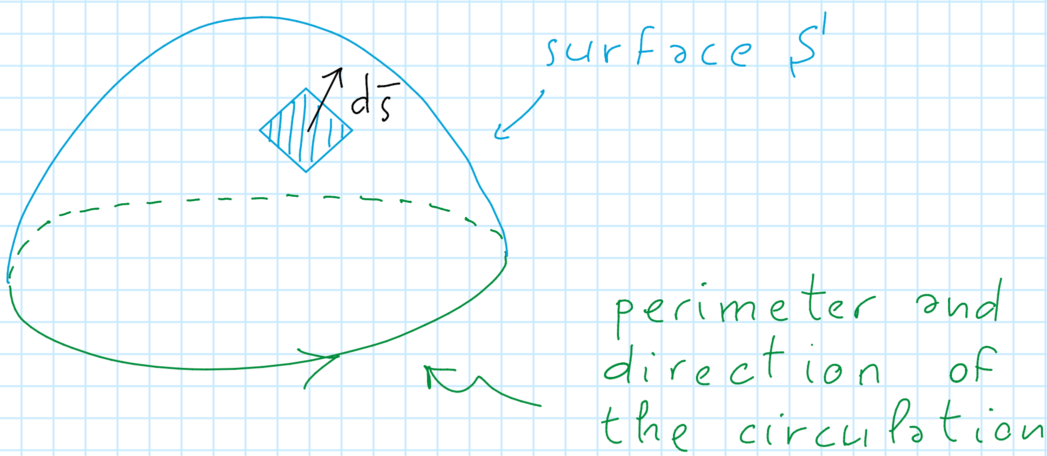
$$\int_S (\nabla \times \bar{v}) \cdot d\bar{s} = \oint_P \bar{v} \cdot d\bar{l}$$

surface

perimeter of the surface

The flux of a curl of a vector through a surface is equal to the circulation of the vector along the line that bounds the surface. The circulation is the line integral of the vector along a closed path.

The direction of the circulation and the direction of the infinitesimal surface element dS are related by the right hand rule.



Corollary 1

The flux of a curl depends only on the boundary line, not on the particular surface chosen

Corollary 2

The flux of a curl through a closed surface is zero, since the boundary line has zero length