Two important mathematical identities

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Here we want to prove two identities which we will repeatedly use in the following

$$\boxed{\nabla \frac{1}{|\bar{r}-\bar{r}'|} = -\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}}$$

$$\boxed{\Delta \frac{1}{|\bar{r}-\bar{r}'|} = -4\pi\,\delta^{(3)}(\bar{r}-\bar{r}')}$$

It is easy to prove the first identity

$$|\bar{r}-\bar{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

Or, using more compact notation

$$|\bar{r}-\bar{r}'| = \sqrt{\sum_{i=1}^{3}(x_i-x_i')^2}$$

$$\nabla \frac{1}{|\bar{r}-\bar{r}'|} = -\frac{1}{|\bar{r}-\bar{r}'|^2}\nabla|\bar{r}-\bar{r}'| = -\frac{1}{|\bar{r}-\bar{r}'|^2}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\sqrt{\sum_{j=1}^{3}(x_j-x_j')^2}\;\underbrace{\hat{\imath}_i}_{\text{UNIT VECTOR IN THE ith DIRECTION}}$$

$$= -\frac{1}{2|\bar{r}-\bar{r}'|^3}\sum_{i=1}^{3}2(x_i-x_i')\,\hat{\imath}_i = -\frac{\bar{r}-\bar{r}'}{|\bar{r}-\bar{r}'|^3}$$

Q.E.D.

Now let's define

$$\bar{r}'' \equiv \bar{r}-\bar{r}' \qquad \text{and} \qquad f \equiv \frac{1}{r''} \qquad \bar{v} \equiv \nabla f = \frac{\partial f}{\partial r''}\hat{r}'' = -\frac{1}{(r'')^2}\hat{r}''$$

One can now calculate the divergence of v by using spherical coordinates

$$\nabla\cdot\bar{v} = \frac{1}{r''}\frac{\partial}{\partial r''}\left((r'')^2\left(-\frac{1}{(r'')^2}\right)\right) = 0 \qquad (\mathrm{I})$$

Let's then calculate the integral of the divergence of v over the volume of a sphere S of radius R

$$\int_S d^3r'' \, \nabla \cdot \bar{v} = \int_{\underbrace{\partial S}_{\text{SURFACE OF THE SPHERE } S}} d\bar{A} \cdot \bar{v} = R^2 \int_{-1}^{1} d\cos\vartheta'' \int_0^{2\pi} d\varphi'' \left(-\frac{1}{R^2}\right) = -4\pi$$

The result above does not depend on the radius of the sphere and it is not consistent with (1). The reason is that the divergence of v is zero almost everywhere:

$$\nabla \cdot \bar{v} = -4\pi \, \delta^{(3)}(\bar{r}'')$$

Now let's observe that the divergence with respect to $\bar{r}''$ is the same as the divergence with respect to $\bar{r}$ if we keep $\bar{r}'$ fixed

$$x_i'' = x_i - x_i' \qquad i = 1, 2, 3 \longrightarrow \frac{\partial}{\partial x_i''} = \frac{\partial}{\partial x_i}$$

Consequently we can write

$$\nabla \cdot \bar{v} = -4\pi \, \delta^{(3)}(\bar{r} - \bar{r}') \qquad \nabla \equiv \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \hat{i}_i$$

Finally, by plugging in the definition of v in terms of f we find

$$\nabla \cdot \bar{v} = \nabla \cdot \nabla f = \Delta \frac{1}{|\bar{r} - \bar{r}'|} = -4\pi \, \delta^{(3)}(\bar{r} - \bar{r}')$$

Q.E.D.