Uniformly polarized sphere

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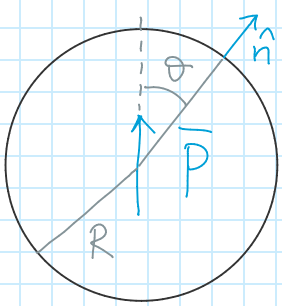
Find the electric field due to a uniformly polarized sphere of radius R.



$$\bar{P} \text{ constant} \rightarrow \nabla \cdot \bar{P} = 0$$

$$\rho_b = -\nabla \cdot \bar{P} = 0$$

$$\sigma_b = \bar{P} \cdot \hat{n} = P \cos\vartheta$$

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \oint_{\partial V} \frac{P\cos\vartheta}{|\bar{r} - \bar{r}'|} \, ds$$

The problem consists in finding the potential due to a sphere carrying a surface charge density P cos ϑ . This problem show azimuthal symmetry, therefore the potential should have an expansion of the form

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty} A_\ell \, r^\ell \, P_\ell(\cos\vartheta) \qquad r \le R$$

$$\varphi(r,\vartheta) = \sum_{\ell=0}^{\infty} B_\ell \, r^{-(\ell+1)} \, P_\ell(\cos\vartheta) \qquad r \ge R$$

The potential must be continuous at r = R, consequently

$$A_\ell R^\ell = B_\ell R^{-(\ell+1)} \rightarrow B_\ell = A_\ell R^{2\ell+1}$$

The electric field (i.e. the radial derivative of the potential) is discontinuous on the surface of the sphere and it should satisfy the condition

$$\left(\frac{\partial}{\partial n} \varphi_{r<R} - \frac{\partial}{\partial n} \varphi_{r>R} \right)_{r=R} = \frac{\sigma}{\varepsilon_0} \qquad \left[\frac{\partial}{\partial n} = \hat{n} \cdot \nabla \right]$$

$$\left.\frac{\partial \varphi_{r<R}}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\vartheta)$$

$$\left.\frac{\partial \varphi_{r>R}}{\partial r}\right|_{r=R} = \sum_{\ell=0}^{\infty} -(\ell+1) B_\ell R^{-(\ell+2)} P_\ell(\cos\vartheta)$$

$$\hookrightarrow \sum_{\ell=0}^{\infty}\left[\ell A_\ell R^{\ell-1} + (\ell+1) B_\ell R^{-(\ell+2)}\right] P_\ell(\cos\vartheta) = \frac{\overbrace{P\cos\vartheta}^{\equiv \sigma_b}}{\varepsilon_0}$$

$$\sum_{\ell=0}^{\infty}\left[\ell A_\ell R^{\ell-1} + (\ell+1) A_\ell R^{\cancel{2}\ell+1-\cancel{\ell}-2}\right] P_\ell(\cos\vartheta) = \frac{P\cos\vartheta}{\varepsilon_0}$$

$$\sum_{\ell=0}^{\infty} A_\ell R^{\ell-1}(2\ell+1) P_\ell(\cos\vartheta) = \frac{P\cos\vartheta}{\varepsilon_0}$$

It is then clear that only the coefficient with l=1 is different from 0

$$A_1\, 3\cos\vartheta = \frac{P\cos\vartheta}{\varepsilon_0} \longrightarrow A_1 = \frac{P}{3\varepsilon_0}$$

$$A_\ell = 0 \quad \text{for } \ell \neq 1$$

Therefore one can conclude that

$$\varphi(r,\vartheta) = \frac{P}{3\varepsilon_0} r\cos\vartheta \qquad \text{for } r \leq R$$

$$\varphi(r,\vartheta) = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\vartheta \qquad \text{for } r \geq R$$

One can then observe that

$$r\cos\vartheta = z$$

The electric field inside the sphere is then constant and directed opposite to the z axis

$$\bar{E} = -\nabla\varphi = -\frac{P}{3\varepsilon_0}\hat{k} = -\frac{\bar{P}}{3\varepsilon_0}$$

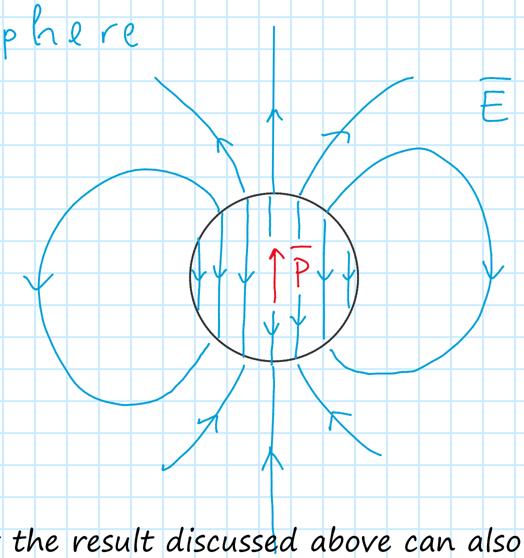
Outside the sphere the potential is the same as the one due to a perfect dipole placed at the center of the sphere

$$\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\bar{p}\cdot\bar{r}}{r^3} = \frac{1}{4\pi\varepsilon_0} \frac{\bar{p}\cdot\hat{n}}{r^2}$$

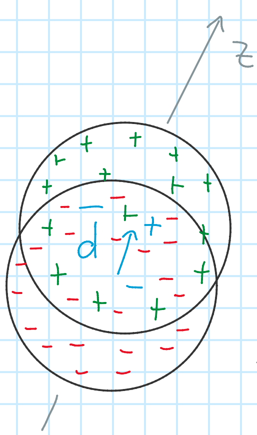
$$\hookrightarrow \bar{p} = \frac{4}{3}\pi R^3 \bar{P}$$

total dipole moment in the sphere

Polarization



Observe that the result discussed above can also be obtained through a very simple model. The polarized sphere can be seen as the overlap of two different spheres, one positively charged and one negatively charged. The polarization arises when the centers of the two spheres are at a short distance apart



$$\bar{E}_+ = \frac{1}{4\pi\varepsilon_0} \frac{q}{R^3} \left(\bar{r} - \frac{d}{2}\hat{k}\right)$$

$$\bar{E}_- = -\frac{1}{4\pi\varepsilon_0} \frac{q}{R^3} \left(\bar{r} + \frac{d}{2}\hat{k}\right)$$

$$\bar{E}_{TOT} = \bar{E}_+ + \bar{E}_- = -\frac{1}{4\pi\varepsilon_0} \frac{q}{R^3} \overbrace{d\hat{k}}^{=\bar{d}}$$

Where E_tot is the total field inside the sphere. One must also have that

$$q\bar{d} = \int_V d^3r\, \bar{P} = \frac{4\pi}{3} R^3\, \bar{P}$$

$$\hookrightarrow \quad \bar{E} = -\frac{1}{\cancel{4\pi}\varepsilon_0}\frac{1}{\cancel{R^3}}\frac{\cancel{4\pi}}{3}\cancel{R^3}\,\bar{P} = -\frac{\bar{P}}{3\varepsilon_0}$$

Which is the same result found with the previous method.