Torque on a magnetic dipole

We saw that the force acting on a magnetic dipole is

$$\bar{F} = -\nabla U \qquad \text{with} \qquad U = -\bar{m}\cdot\bar{B}$$

$$\bar{F} = \nabla(\bar{m}\cdot\bar{B}) = \bar{m}\times(\nabla\times\bar{B}) + \bar{B}\times(\nabla\times\bar{m}) + (\bar{m}\cdot\nabla)\bar{B}$$

$$+ (\bar{B}\cdot\nabla)\bar{m}$$

$$= \bar{m}\times\underbrace{(\nabla\times\bar{B})}_{=0 \ \text{if} \ \bar{j}=0} + (\bar{m}\cdot\nabla)\bar{B} = (\bar{m}\cdot\nabla)\bar{B}$$

The force is then zero if B is constant.

$$\left[\begin{array}{c}\text{rem electric dipole}\\ \bar{F} = (\bar{p}\cdot\nabla)\bar{E}\end{array}\right]$$

Here we want to determine the torque acting on the magnetic dipole. One can start from the vector potential due to a dipole placed at x_0.

$$\bar{A}(\bar{x}) = \frac{\mu_o}{4\pi}\ \frac{\bar{m}\times(\bar{x}-\bar{x}_o)}{|\bar{x}-\bar{x}_o|^3} \qquad\longrightarrow\qquad \bar{B} = \nabla\times\bar{A}$$

rem

$$\nabla\times(\bar{a}\times\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b} + \bar{a}(\nabla\cdot\bar{b}) - \bar{b}(\nabla\cdot\bar{a})$$

$$\nabla\times\left(\bar{m}\times\frac{(\bar{x}-\bar{x}_o)}{|\bar{x}-\bar{x}_o|^3}\right) = -(\bar{m}\cdot\nabla)\left(\frac{\bar{x}-\bar{x}_o}{|\bar{x}-\bar{x}_o|^3}\right) + \bar{m}\ \ \nabla\cdot\left(\frac{\bar{x}-\bar{x}_o}{|\bar{x}-\bar{x}_o|^3}\right)$$

$$\bar{B}(\bar{x}) = \frac{\mu_o}{4\pi}\left[\bar{m}\ \nabla\cdot\frac{\bar{x}-\bar{x}_o}{|\bar{x}-\bar{x}_o|^3} - (\bar{m}\cdot\nabla)\frac{\bar{x}-\bar{x}_o}{|\bar{x}-\bar{x}_o|^3}\right]$$

$$= \frac{\mu_o}{4\pi}\left[-\bar{m}\ \nabla\cdot\left(\nabla\frac{1}{|\bar{x}-\bar{x}_o|}\right) - (\bar{m}\cdot\nabla)\frac{\bar{x}-\bar{x}_o}{|\bar{x}-\bar{x}_o|^3}\right]$$

rem

$$\nabla(\bar{a}\cdot\bar{b}) = \bar{a}\cdot\nabla\bar{b} + \bar{b}\cdot\nabla\bar{a}$$

$$\bar{B}(\bar{x}) = \frac{\mu_o}{4\pi}\left[-\bar{m}\ \triangle\frac{1}{|\bar{x}-\bar{x}_o|} - \nabla\left(\frac{\bar{m}\cdot(\bar{x}-\bar{x}_o)}{|\bar{x}-\bar{x}_o|^3}\right)\right]$$

From Ampere's equation in magnetostatics one has

$$\nabla \times \overline{B} = \mu_0 \overline{J} \longrightarrow \overline{J} = \frac{1}{\mu_0} \nabla \times \overline{B}$$

$$\overline{J}(\overline{x}) = \frac{1}{4\pi}\left[-\nabla \times \left(\overline{m} \; \underbrace{\Delta \frac{1}{|\overline{x} - \overline{x}_0|}}_{-4\pi \delta^{(3)}(\overline{x}-\overline{x}_0)} \right) - \underbrace{\nabla \times \nabla \frac{\overline{m}\cdot(\overline{x}-\overline{x}_0)}{|\overline{x}-\overline{x}_0|^3}}_{=\,0} \right]$$

$$= \nabla \times \left(\overline{m} \; \delta^{(3)}(\overline{x}-\overline{x}_0) \right)$$

<u>rem</u> $\quad \nabla \times (f\,\overline{A}) = f\,\nabla \times \overline{A} - \overline{A} \times \nabla f$

$$\overline{J}(\overline{x}) = -\,\overline{m} \times \nabla \delta^{(3)}(\overline{x}-\overline{x}_0) \qquad$$ CURRENT DENSITY OF A MAGNETIC DIPOLE

The torque that we want to evaluate is

$$\overline{T} = \int \overline{x} \times d\overline{F} = \int d^3x \; \overline{x} \times (\overline{J} \times \overline{B}) \qquad$$ external B, not the B due to $\overline{J}$

$$= -\int d^3x \; \overline{x} \times \left[\left(\overline{m} \times \nabla \delta^{(3)}(\overline{x}-\overline{x}_0) \right) \times \overline{B}(\overline{x}) \right]$$

$$= \int d^3x \; \overline{x} \times \left[\overline{B}(\overline{x}) \times \left(\overline{m} \times \nabla \delta^{(3)}(\overline{x}-\overline{x}_0) \right) \right]$$

<u>rem</u>

$$\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a}\cdot\overline{c})\,\overline{b} - (\overline{a}\cdot\overline{b})\,\overline{c}$$

$$\overline{T} = \int d^3x \; \overline{x} \times \left[\left(\overline{B}\cdot\nabla \delta^{(3)}(\overline{x}-\overline{x}_0) \right)\overline{m} - \left(\overline{B}\cdot\overline{m} \right) \nabla \delta^{(3)}(\overline{x}-\overline{x}_0) \right]$$

Consider now a component of the torque

$$T_i = \int d^3x \; \varepsilon_{ijk}\, x_j \left[B_\ell\, \partial_\ell\, \delta^{(3)}(\overline{x}-\overline{x}_0)\, m_k - m_\ell\, B_\ell\, \partial_k\, \delta^{(3)}(\overline{x}-\overline{x}_0) \right]$$

$$= -\int d^3x \; \varepsilon_{ijk}\, \partial_\ell (x_j\, B_\ell)\, \delta^{(3)}(\overline{x}-\overline{x}_0)\, m_k \qquad$$ BY PARTS

$$+ \int d^3x \; \varepsilon_{ijk}\, \partial_k (x_j\, B_\ell)\, \delta^{(3)}(\overline{x}-\overline{x}_0)\, m_\ell$$

In the integration by parts, the surface terms vanish since we are integrating over R^3 and the delta function is zero everywhere except for x_O.

$$T_i = \int d^3x \; \delta^{(3)}(\bar{x}-\bar{x}_o) \; \varepsilon_{ijk} \left[m_\ell \partial_k - m_k \partial_\ell \right] x_j B_\ell$$

$$= \int d^3x \; \delta^{(3)}(\bar{x}-\bar{x}_o) \; \varepsilon_{ijk} \left[m_\ell \underset{\nearrow^0 \quad \varepsilon_{ijk}\delta_{jk}=0}{\delta_{jk} B_\ell} + m_\ell x_j \partial_k B_\ell \right.$$

$$\left. - m_k \delta_{\ell j} B_\ell - m_k x_j \underset{\nearrow^0}{\partial_\ell B_\ell} \right]$$

$$\underbrace{\qquad}_{\nabla \cdot \bar{B} = 0}$$

$$= \int d^3x \; \delta^{(3)}(\bar{x}-\bar{x}_o) \; \varepsilon_{ijk} \; x_j \underset{\overset{\partial_k(\bar{m}\cdot\bar{B})}{\nearrow}}{(\bar{m}\cdot\partial_k \bar{B})} - \int d^3x \; \delta^{(3)}(\bar{x}-\bar{x}_o) \; \varepsilon_{ijk} \; m_k B_j$$

$$= \left[\bar{m} \times \bar{B}(\bar{x}_o) \right]_i + \left[\bar{x}_o \times \nabla(\bar{m}\cdot\bar{B}) \right]_i$$

We saw that the force acting on a dipole is

$$\bar{F} = -\nabla(-\bar{m}\cdot\bar{B}) = \nabla(\bar{m}\cdot\bar{B})$$

$$\hookrightarrow \bar{x}_o \times \nabla(\bar{m}\cdot\bar{B}) = \bar{x}_o \times \bar{F}(\bar{x}_o)$$

So that finally

$$\bar{T} = \bar{m} \times \bar{B}(\bar{x}_o) + \bar{x}_o \times \bar{F}(\bar{x}_o)$$

Consequently if the dipole is located at the origin of the frame of reference one finds

$$\bar{x}_o = 0 \longrightarrow \boxed{\bar{T} = \bar{m} \times \bar{B}}$$

TORQUE ACTING ON A MAGNETIC DIPOLE

$$\left[\text{rem} \quad \text{electric dipole} \atop \bar{T} = \bar{p} \times \bar{E} \right]$$