Product rules

Tuesday, February 1, 2022 11:36 AM

G problem 25

Consider the vectors

$$\bar{A} = x\hat{i} + 2y\hat{j} + 3z\hat{k} \qquad \bar{B} = 3y\hat{i} - 2x\hat{j}$$

Check that

$$\nabla\cdot(\bar{A}\times\bar{B}) = \bar{B}\cdot(\nabla\times\bar{A}) - \bar{A}\cdot(\nabla\times\bar{B})$$

$$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & 2y & 3z \\ 3y & -2x & 0 \end{vmatrix} = +6xz\hat{i} + 9yz\hat{j} - (2x^2+6y^2)\hat{k}$$

$$\nabla\cdot(\bar{A}\times\bar{B}) = 6z + 9z = \boxed{15z}$$

$$\nabla\times\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & 2y & 3z \end{vmatrix} = 0$$

$$\nabla\times\bar{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3y & -2x & 0 \end{vmatrix} = \hat{k}(-2-3) = -5\hat{k}$$

$$-\bar{A}\cdot(\nabla\times\bar{B}) = -3z\times(-5) = \boxed{15z} \quad \checkmark$$

Check that

$$\nabla(\bar{A}\cdot\bar{B}) = \bar{A}\times(\nabla\times\bar{B}) + \bar{B}\times(\nabla\times\bar{A}) + (\bar{A}\cdot\nabla)\bar{B} + (\bar{B}\cdot\nabla)\bar{A}$$

$$= \bar{A}\times(\nabla\times\bar{B}) + (\bar{A}\cdot\nabla)\bar{B} + (\bar{B}\cdot\nabla)\bar{A}$$

Since we already know that $\nabla\times\bar{A} = 0$

$$\bar{A}\cdot\bar{B} = 3xy - 4xy = -xy$$

$$\nabla(\bar{A}\cdot\bar{B}) = \boxed{-y\hat{i} - x\hat{i}}$$

$$\bar{A}\times(\nabla\times\bar{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & 2y & 3z \\ 0 & 0 & -5 \end{vmatrix} = -10y\,\hat{i} + 5x\,\hat{j}$$

$$(\bar{A}\cdot\nabla)\bar{B} = (x\partial_x + 2y\partial_y + 3z\partial_z)(3y\,\hat{i} - 2x\,\hat{j})$$

$$= -2x\,\hat{j} + 6y\,\hat{i}$$

$$(\bar{B}\cdot\nabla)\bar{A} = (3y\partial_x - 2x\partial_y)(x\,\hat{i} + 2y\,\hat{j} + 3z\,\hat{k})$$

$$= 3y\,\hat{i} - 4x\,\hat{j}$$

$$\bar{A}\times(\nabla\times\bar{B}) + (\bar{A}\cdot\nabla)\bar{B} + \bar{B}(\nabla\cdot\bar{A}) =$$

$$= -10y\,\hat{i} + 5x\,\hat{j} - 2x\,\hat{j} + 6y\,\hat{i} + 3y\,\hat{i} - 4x\,\hat{j}$$

$$= \boxed{-y\,\hat{i} - x\,\hat{j}} \quad \checkmark$$

Check that

$$\nabla \times (\bar{A} \times \bar{B}) = (\bar{B} \cdot \nabla)\bar{A} - (\bar{A} \cdot \nabla)\bar{B} + \bar{A}(\nabla \cdot \bar{B}) - \bar{B}(\nabla \cdot \bar{A})$$

$$\nabla \times (\bar{A} \times \bar{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 6xz & 9yz & -(2x^2 + 6y^2) \end{vmatrix}$$

$$= \hat{i}(-12y - 9y) - \hat{j}(-4x - 6x) = \boxed{-21y\,\hat{i} + 10x\,\hat{i}}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \cdot \bar{A} = 6$$

$$(\bar{B} \cdot \nabla)\bar{A} - (\bar{A} \cdot \nabla)\bar{B} - \bar{B}(\nabla \cdot \bar{A})$$

$$= 3y\,\hat{i} - 4x\,\hat{j} + 2x\,\hat{j} - 6y\,\hat{i} - 18y\,\hat{i} + 12x\,\hat{j}$$

$$= \hat{i}\,y(3 - 6 - 18) + \hat{j}\,x(-4 + 2 + 12)$$

$$= \boxed{-21y\,\hat{i} + 10x\,\hat{j}} \quad \checkmark$$