Charges and currents

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In this course we use the SI system of units

There are 4 fundamental forces in nature:

1) Gravity, binds stars, responsible for the motion of the planets in the solar system, creates tides
2) Strong force, binds quarks in barions and mesons
3) Weak forces, mediates interactions involving neutrinos, changes the flavor of quarks, responsible for beta decay
4) Electromagnetism, responsible for the behavior of charged particles in electric and magnetic fields

The basic force which regulates the behavior of a charge in an electric and magnetic field is the Coulomb-Lorentz force

$$\bar{F} = q\left(\bar{E} + \bar{v}\times\bar{B}\right)$$

We call electromagnetism the study of the origin and behavior of the electric and magnetic fields $\bar{E}(\bar{r},t)$ and $\bar{B}(\bar{r},t)$

Electric Charge

Electric charge is a variable which can be assigned to a body. It is a positive or negative integer multiple of the proton charge *

$$e = 1.6021733(49) \times 10^{-19} \text{ Coulomb}$$

With this notation the electron has a charge $-e$

(* quarks have fractionary charges $+\frac{2}{3}e$ and $-\frac{1}{3}e$, but they are confined)

The quantum of charge is "very small", so it is convenient to define a continuous charge per unit volume, or volume charge density $\rho(\bar{r})$

Therefore an infinitesimal and finite charge can be written as

$$dQ = \rho(\bar{r})\, d^3r \quad ; \quad Q = \int_V \rho(\bar{r})\, d^3r \qquad [\rho] = \frac{C}{m^3}$$

Example: Charge in a sphere with uniform charge density

$$\rho(\bar{r}) \equiv \rho_0 \to \text{const} \qquad Q = \rho_0 \int_V d^3r = \rho_0 \int_0^R dr\, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi = \frac{4\pi}{3} R^3 \rho_0$$

It is often necessary to work with charge distributed on an infinitesimally thin layer. In this case one defines a surface charge density $\sigma(\bar{r}_s)$

$[\sigma] = \frac{C}{m^2}$

$\bar{r}_s \to$ point on the surface S

In this case

$$dQ = \sigma(\bar{r}_s)\, ds \quad ; \qquad Q = \int_S \sigma(\bar{r}_s)\, ds$$

Example: Charge uniformly distributed over the surface of a sphere

$$\sigma(\bar{r}_s) \equiv \sigma_0 \to \text{const} \qquad Q = \sigma_0 \int R^2\, d\cos\vartheta\, d\varphi = 4\pi R^2 \sigma_0$$

For charges distributed in one dimension one can define line charge density $\lambda(\bar{r}_c)$

$\bar{r}_c \to$ point on the line c

$[\lambda] = \frac{C}{m}$

Consequently

$$dQ = \lambda(\bar{r}_c)\, dl \quad ; \qquad Q = \int_c \lambda(\bar{r}_c)\, dl$$

In principle since fundamental particles are point-like objects it is possible (although not always useful) to write a charge distribution as

$$\rho(\bar{r}) = \sum_{k=1}^{N} q_k\, \delta^{(3)}(\bar{r} - \bar{r}_k) \qquad \left(N = \text{total number of particles}\right)$$

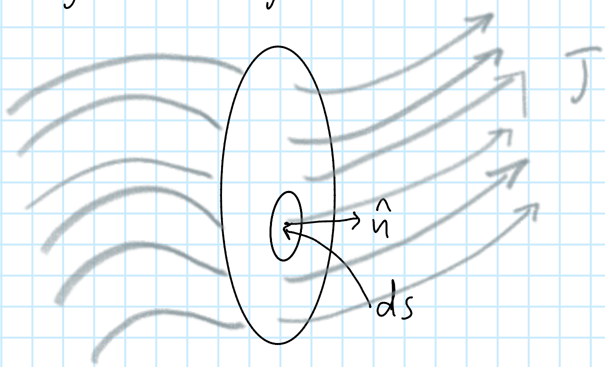
Electric Current

An electric current is a flow of electric charges. We describe it in analogy with a fluid flow. For this purpose we need to define a current density $\bar{J}(\bar{r}, t)$

$$[\bar{J}] = \frac{C}{m^2 s}$$

$d\bar{S} \equiv dS\,\hat{n}$ $\qquad \hat{n}$ = normal to the infinitesimal surface element dS

$$dI \equiv \bar{J} \cdot d\bar{S} = \bar{J} \cdot \hat{n}\, dS$$



Electric current

$$I \equiv \frac{dQ}{dt} = \int_S d\bar{S} \cdot \bar{J}$$

$$[I] = \frac{C}{s}$$

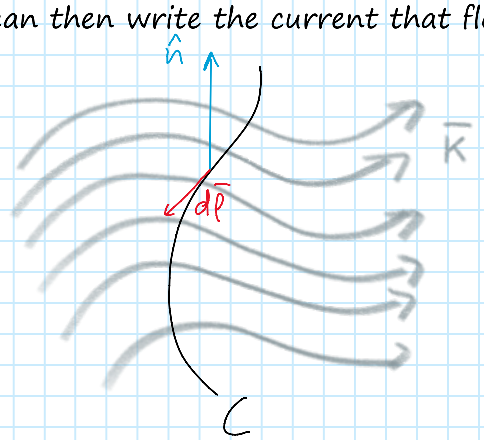
One can write a current density as a function of the charge density and of the velocity of each point of the charge density:

$$\bar{J}(\bar{r}, t) = \rho(\bar{r}, t)\, \bar{v}(\bar{r}, t)$$

If the moving charges are confined to a surface one needs to use a surface current density $\bar{K}$

$$\bar{K}(\bar{r}_s, t) \equiv \sigma(\bar{r}_s, t)\, \bar{v}(\bar{r}_s, t)$$

One can then write the current that flows past a curve C on the surface as follows



$$I = \int_C d\bar{\ell} \cdot (\bar{K} \times \hat{n}) = \int_C \bar{K} \cdot (\hat{n} \times d\bar{\ell})$$

$\hat{n}$ = normal to the surface where $\bar{K}$ flows

Conservation of charge and continuity equation

Charge is conserved in all known physical processes. A local statement of the conservation of charge is the continuity equation, which can be obtained as follows. Assume that S is a closed surface which encloses the volume V

$$I = \int_S d\bar{S} \cdot \bar{J} \underset{\uparrow \atop \text{div. theorem}}{=} \int_V d^3r \, \nabla \cdot \bar{J} = -\underbrace{\frac{dQ}{dt}}_{\substack{\text{rate at which} \\ \text{the charge in V changes}}} = -\frac{d}{dt}\int_V d^3r \, \rho = -\int_V d^3r \, \frac{\partial}{\partial t}\rho$$

The minus sign is motivated by the fact that the normal to the volume, which determines the sign of the flux integral, points out of the volume. One can then conclude that

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \bar{J} = 0} \qquad \text{Continuity equation}$$