Conservative force

Tuesday, January 4, 2022 10:17 AM

Coulomb's force is conservative: The work done by the electrostatic force on a test charge when the charge moves from A to B depends only on the value of the potential at the points A and B.

$$W = \int_{\bar{r}_A}^{\bar{r}_B} d\bar{\ell} \cdot \bar{F} = q \int_{\bar{r}_A}^{\bar{r}_B} d\bar{\ell} \cdot \bar{E} = -q \int_{\bar{r}_A}^{\bar{r}_B} d\bar{\ell} \cdot \nabla\varphi$$

test charge

$$= -q \int_{\bar{r}_A}^{\bar{r}_B} \left(\frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy + \frac{\partial\varphi}{\partial z} dz \right) = (*)$$

$$\int_{\bar{r}_A}^{\bar{r}_B} = \int_{\{x_A, y_A, z_A\}}^{\{x_B, y_A, z_A\}} + \int_{\{x_B, y_A, z_A\}}^{\{x_B, y_B, z_A\}} + \int_{\{x_B, y_B, z_A\}}^{\{x_B, y_B, z_B\}}$$

$$W = (*) = -q \left\{ \int_{\{x_A, y_A, z_A\}}^{\{x_B, y_A, z_A\}} dx \frac{\partial\varphi}{\partial x} + \int_{\{x_B, y_A, z_A\}}^{\{x_B, y_B, z_A\}} dy \frac{\partial\varphi}{\partial y} + \int_{\{x_B, y_B, z_A\}}^{\{x_B, y_B, z_B\}} dz \frac{\partial\varphi}{\partial z} \right\}$$

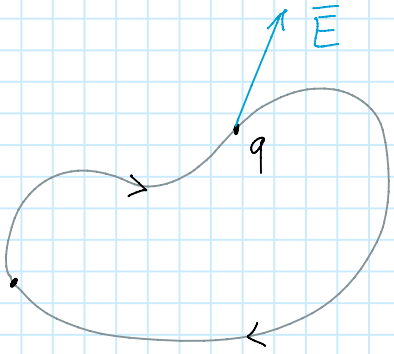
$$= -q \left\{ \varphi(x_B, y_A, z_A) - \varphi(x_A, y_A, z_A) + \varphi(x_B, y_B, z_A) - \varphi(x_B, y_A, z_A) \right.$$

$$\left. + \varphi(x_B, y_B, z_B) - \varphi(x_B, y_B, z_A) \right\}$$

$$= -q \left[\varphi(x_B, y_B, z_B) - \varphi(x_A, y_A, z_A) \right]$$

The result above is independent from the path chosen (As one can see if one evaluates the integral over any other sequence of points connecting A to B)

By using Stokes's theorem one can directly prove that the work done by Coulomb's force along a closed path is zero. This fact is itself a proof of the fact that the electrostatic force is conservative.



$$W = q \oint d\bar{\ell} \cdot \bar{E} = q \int_S d\bar{s}\, \underbrace{\nabla \times \bar{E}}_{=0} = 0$$

in electrostatics

Going back to the integral at the beginning of this set of notes, one can see that the potential difference between two points can be determined by evaluating an integral:

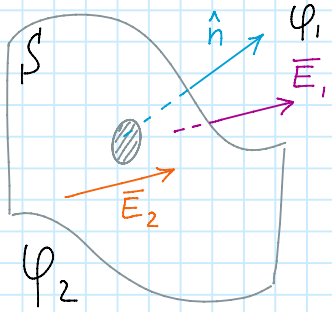
$$\varphi(\bar{r}) - \varphi(\bar{r}_0) = -\int_{\bar{r}_0}^{\bar{r}} d\bar{\ell} \cdot \bar{E}$$

When one sets to zero the potential at the initial integration point (typically a point infinitely far away from the charge distribution) the expression above simplifies to

$$\varphi(\bar{r}) = -\int_{\bar{r}_0}^{\bar{r}} d\bar{\ell} \cdot \bar{E}$$

Matching conditions for the potential

The matching conditions for E on the two sides of a surface with a surface charge density were found to be



$$\hat{n} \cdot (\bar{E}_1 - \bar{E}_2) = \frac{\sigma}{\varepsilon_0}$$

THE COMPONENT OF $\bar{E}$ PERPENDICULAR TO THE SURFACE IS DISCONTINUOUS

This implies that

$$\hat{n}\cdot\nabla\varphi_2 - \hat{n}\cdot\nabla\varphi_1 = \frac{\sigma}{\varepsilon_0}$$

$$\hat{n}\cdot\nabla \equiv \frac{\partial}{\partial n}$$

DERIVATIVE IN THE NORMAL DIRECTION

$$\left(\frac{\partial\varphi_2}{\partial n} - \frac{\partial\varphi_1}{\partial n}\right)_{\text{at the surface}} = \frac{\sigma}{\varepsilon_0}$$

Notice however that the potential is continuous at the surface, since each point in space must have a single value for the potential

$$\left(\varphi_1 - \varphi_2\right)_{\text{at the surface}} = 0$$

If one wants to use a mathematically more precise notation, one should write

$$\lim_{\varepsilon\to 0}\left(\varphi_1(\bar{r}_s+\varepsilon\hat{n}) - \varphi_2(\bar{r}_s-\varepsilon\hat{n})\right) = 0$$

$\bar{r}_s$ is a point on the surface S