Electrostatic potential

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Remember the electrostatics equations

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0}$$ GAUSS' EQUATION FOR $\bar{E}$

$$\nabla \times \bar{E} = 0$$ $\bar{E}$ HAS A ZERO CURL EVERYWHERE

The second equation implies that electric field is the gradient of a scalar function, which is called scalar potential or electrostatic potential (or simply potential when there are no ambiguities)

$$\nabla \times \bar{E} = 0 \longrightarrow \bar{E} = -\nabla \varphi$$ (the minus sign is a convention)

The potential is defined up to an additive constant

$$\bar{E} = -\nabla \varphi = -\nabla(\varphi + \varphi_0)$$ constant

One can then rewrite Gauss' law in terms of the potential, in this way one finds Poisson's equation

$$\nabla \cdot \bar{E} = \nabla \cdot (-\nabla \varphi) = -\nabla \cdot \nabla \varphi = -\Delta \varphi = \frac{\rho}{\varepsilon_0}$$

$$\boxed{\Delta \varphi = -\frac{\rho}{\varepsilon_0}}$$ POISSON'S EQUATION

Formal solution of Poisson's equation

Poisson's equation has a useful solution for a charge distribution localized in a finite volume. In this solution the constant ambiguity in the potential is fixed by requiring that the potential vanishes at infinity far away from the source charges.

$$\boxed{\varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}}$$

The equation above is indeed the solution of Poisson's equation since

$$\Delta_{\bar{r}}\, \varphi(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \rho(\bar{r}') \underbrace{\Delta_{\bar{r}} \frac{1}{|\bar{r}-\bar{r}'|}}_{-4\pi\,\delta^{(3)}(\bar{r}-\bar{r}')}$$

$$= -\frac{1}{\varepsilon_0} \int d^3r' \, \rho(\bar{r}')\, \delta^{(3)}(\bar{r}-\bar{r}')$$

$$= -\frac{\rho(\bar{r})}{\varepsilon_0}$$

Q.E.D.

This is formally all what we need to study electrostatic situations with charges that are confined in a finite volume. Unfortunately, the evaluation of the integral needed to determine the potential can in general be very complicated.