Fourier Series

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A complete set of functions in the interval [0,a] is given by

$$g_n(x) = \sin\left(\frac{n\pi}{a}x\right)$$

The functions are orthogonal

$$\int_0^a dx \ \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) = 0 \qquad \text{if } n \neq m$$

Proof

$$\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\int_0^a dx \ \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) =$$

$$= \frac{1}{2}\left\{\int_0^a dx \ \cos\left[(n-m)\frac{\pi}{a}x\right] - \int_0^a dx \ \cos\left[(n+m)\frac{\pi}{a}x\right]\right\}$$

$$= \frac{1}{2}\frac{a}{(n-m)\pi}\int_0^{(n-m)\pi} du \ \cos u - \frac{1}{2}\frac{a}{(n+m)\pi}\int_0^{(n+m)\pi} du \ \cos u$$

$$= \frac{a}{2\pi}\left(\frac{1}{(n-m)} \sin u \Big|_0^{(n-m)\pi} - \frac{1}{n+m} \sin u \Big|_0^{(n+m)\pi}\right) = 0$$

q.e.d.

However the functions are not yet normalized, in fact

$$\int_0^a dx \ \sin^2\left(\frac{n\pi}{a}x\right) = \frac{a}{n\pi}\int_0^{n\pi} du \ \sin^2 u = \frac{a}{n\pi}\frac{1}{2}n\pi = \frac{a}{2}$$

for $n \neq 0$

and $\int_0^a dx \, \sin^2\left(0 \frac{\pi}{a} x\right) = 0$ for $n = 0$

Therefore an orthonormal set of functions will be

$$f_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

These functions satisfy the completeness relation

$$\boxed{\sum_{n=1}^{\infty} f_n(x) f_n(x') = \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} x'\right) = \delta(x - x')}$$

This relation is not easy to prove. A possible way to look at it is to integrate both sides between x' = 0 and x'= x_0 to obtain

$$\theta(x - x_0) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - \cos\left(\frac{n\pi x}{a}\right)\right] \sin\left(\frac{n\pi x_0}{a}\right)$$

and then draw the sum on the rhs to see that it approaches a step function.

We saw that a function in the interval [0,a] can be written in terms of the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a} x\right)$$

If one extends the domain of the function to [-a,a], the resulting function will be odd, since the series involves only sine functions, which are odd

$$f(-x) = \sum_{n=1}^{\infty} b_n \sin\left(-\frac{n\pi}{a} x\right) = -\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a} x\right) = -f(x)$$

It is possible to write a function in the interval [0,a] in terms of even functions, so that when the domain is extended to [-a,a] the resulting function is even. This series is called Fourier cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a} x\right)$$

A generic function in the interval [-a,a] is neither even nor odd, and any function can be written as the sum of an even and an odd part. Therefore one can describe a function in the interval [-a,a] with the Fourier trigonometric series, that is a combination of the sine and the cosine series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{a} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a} x\right)$$

Therefore the set of orthonormal functions in the interval [-a/2,a/2], of the same length as the original interval that we considered is

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{a}} \\ \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a} n\right) \\ \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi x}{a} n\right) \end{cases} \qquad n = 1, 2, 3, \ldots$$

One can then observe that

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad \cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

The above should make it clear that another complete set of functions over the interval [-a/2,a/2] is

$$U_m(x) = \frac{1}{\sqrt{a}} e^{i\left(\frac{2\pi x}{a} m\right)} \qquad m = 0, \pm 1, \pm 2, \pm 3 \ldots$$

Problem: Check that the functions are correctly normalized, i.e.

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \; U_m^*(x)\, U_n(x) = \delta_{nm}$$

A generic function f can therefore be written as

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=-\infty}^{+\infty} A_m \exp\left(\frac{2\pi i x}{a} m\right)$$

$$A_m = \frac{1}{\sqrt{a}} \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \; \exp\left(-\frac{2\pi i x}{a} m\right) f(x)$$

At this stage we can take the continuum limit (i.e. send a to infinity)

$$a \to \infty \qquad \frac{2\pi m}{a} \to k \qquad \sum_m \to \frac{a}{2\pi} \int_{-\infty}^{+\infty}$$

$$A_m \to \sqrt{\frac{2\pi}{a}}\, A(k)$$

factor needed to fix normalization

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \; A(k)\, e^{ikx} \\ A(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \; f(x)\, e^{-ikx} \end{aligned}}$$

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In taking the limit one traded a numerable parameter m with a continuous parameter k (k can be a quantity with physical dimensions). The orthonormal set of functions parameterized by k is therefore

$$U(k) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

In this case, the orthogonality and completeness relations show complete symmetry in the exchange x <--> k

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \; e^{i(k-k')x} = \delta(k-k')$$ ORTHOGONALITY

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{ik(x-x')} = \delta(x-x')$$ COMPLETENESS