Wave equation

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Starting from the complete Maxwell's equations in the special case in which one looks at a portion of space where there are no sources, it is possible to show that the electric and magnetic field satisfy wave equations. When there are no sources, Maxwell's equations become

$$\nabla \cdot \overline{E} = 0 \qquad \nabla \cdot \overline{B} = 0$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \nabla \times \overline{B} = \mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

A time variation in B produces a (varying) field E; a time variation in E produces a field B. Now one can take the time derivative of the equation involving the curl of B

$$\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2} = \frac{\partial}{\partial t}\left(\nabla \times \overline{B}\right) = \nabla \times \frac{\partial \overline{B}}{\partial t}$$

The time derivative of B can be obtained from the equation involving the curl of E

$$\mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2} = \nabla \times \left(-\nabla \times \overline{E}\right)$$

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$$\nabla \times \left(\nabla \times \overline{E}\right) = \nabla\left(\nabla \cdot \overline{E}\right) - \Delta \overline{E}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2} = -\nabla\underbrace{\left(\nabla \cdot \overline{E}\right)}_{=0} + \Delta \overline{E}$$

$$\underbrace{\mu_0 \varepsilon_0}_{=\frac{1}{c^2}} \frac{\partial^2 \overline{E}}{\partial t^2} - \Delta \overline{E} = 0$$

c = SPEED OF THE WAVE

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} - \Delta \overline{E} = 0}$$

WAVE EQUATION

Also B satisfies a wave equation. In fact, one finds

$$\frac{\partial^2 \overline{B}}{\partial t^2} = -\frac{\partial}{\partial t}\left(\nabla\times\overline{E}\right) = -\nabla\times\frac{\partial \overline{E}}{\partial t} = -\frac{1}{\mu_0 \varepsilon_0}\nabla\times\left(\nabla\times\overline{B}\right)$$

$$= -\frac{1}{\mu_0 \varepsilon_0}\left(\nabla\underbrace{\left(\nabla\cdot\overline{B}\right)}_{=0} - \Delta\overline{B}\right) = \frac{1}{\mu_0 \varepsilon_0}\Delta\overline{B}$$

$$\boxed{\frac{1}{c^2}\frac{\partial^2 \overline{B}}{\partial t^2} - \Delta\overline{B} = 0}$$

The value of c can be figured out by plugging in the values of the vacuum permittivity and permeability

$$\varepsilon_0 = 8.854187817 \times 10^{-12}\ \frac{s^2 C^2}{m^3\, Kg}$$

$$\mu_0 = 4\pi \times 10^{-7}\ \frac{m\, Kg}{C^2}$$

$$\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 299{,}792{,}458\ \frac{m}{s}}$$

When Maxwell calculated the velocity of propagation of the electromagnetic waves, he noticed that it was close to the experimental value of the speed of light. He then advanced the hypothesis that light is an electromagnetic wave. This is indeed the case, and visible light is just a small part of the frequency spectrum of electromagnetic waves.