Uniformly magnetized sphere

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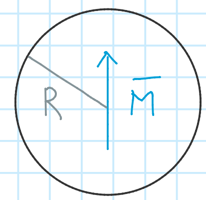
(from Jackson 5.10)

Here we study the magnetic field created by uniformly magnetized sphere. We use the method of the scalar magnetic potential. One can apply this method because there are no free currents. Let's choose the z axis in the direction of the magnetization

$$\bar{M} = M_0 \hat{k}$$



In order to discuss appropriately this case we need to consider carefully the possible presence of a term induced by possible "surface current density" running on the surface of the sphere

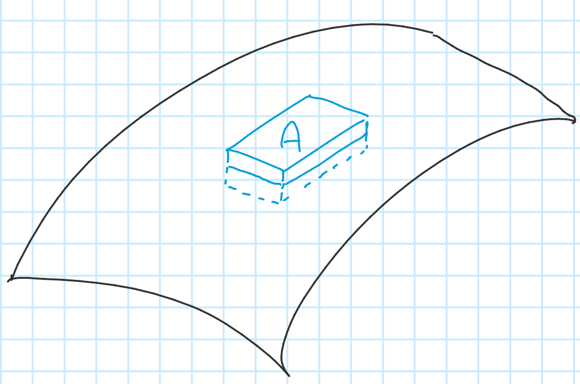
$$\bar{J}_f = 0 \implies \nabla \times \bar{H} = 0 \implies \bar{H} = -\nabla \varphi_M$$

$$\bar{H} = \frac{1}{\mu_0}\bar{B} - \bar{M} \implies \nabla \cdot \bar{B} = \mu_0 \nabla \cdot (\bar{H} + \bar{M}) = 0$$

$$\implies \mu_0 \left(-\nabla \cdot \nabla \varphi_M + \nabla \cdot \bar{M} \right) = 0 \implies \Delta \varphi_M = \underbrace{+\nabla \cdot \bar{M}}_{\equiv -\rho_M}$$

$$\varphi_M(\bar{x}) = -\frac{1}{4\pi}\int_V d^3y \, \frac{\nabla_{\bar{y}} \cdot \bar{M}(\bar{y})}{|\bar{x} - \bar{y}|} = +\frac{1}{4\pi}\int_V d^3y \, \frac{\rho_M}{|\bar{x} - \bar{y}|}$$

Now we want to separate the integral on the interior of the sphere from the integral on the surface of the sphere. Consider a pillbox volume of vanishing thickness over the surface of the sphere.



The volume integral of rho_M over the pillbox volume is

$$\int_V d^3y\, \rho_M = -\int_V \nabla\cdot\overline{M}\, d^3y = -\oint_{\partial V} \overline{M}\cdot\hat{n}\, dA$$

$$\hookrightarrow \quad \sigma_M \equiv -\overline{M}\cdot\hat{n}$$

in analogy with the electrostatic case

Therefore one can rewrite the magnetic scalar potential as

$$\varphi_M(\overline{x}) = -\frac{1}{4\pi}\int_V d^3y\, \frac{\nabla_{\overline{y}}\cdot\overline{M}(\overline{y})}{|\overline{x}-\overline{y}|} + \frac{1}{4\pi}\oint_{\partial V} \frac{\overline{M}\cdot\hat{n}}{|\overline{x}-\overline{y}|}\, dA$$

Since the magnetization of the sphere is constant, its divergence is zero. Only the surface integral contributes to the scalar magnetic potential.

$$\varphi_M(\overline{x}) = \frac{1}{4\pi}\oint \frac{\overline{M}\cdot\hat{n}}{|\overline{x}-\overline{y}|}\, dA$$

In spherical coordinates

$$\varphi_M(r,\vartheta) = \frac{M_0 R^2}{4\pi}\int d\cos\vartheta'\, d\phi'\, \frac{\cos\vartheta'}{|\overline{x}-\overline{y}|}$$

$\overline{y} \equiv \{R, \vartheta', \phi'\}$
point on the surface of the sphere

One can then use the identity

$$\boxed{\frac{1}{|\overline{x}-\overline{y}|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}Y^*_{\ell m}(\vartheta',\phi')\,Y_{\ell m}(\vartheta,\phi)\,\frac{r_<^{\ell}}{r_>^{\ell+1}}}$$

WARNING: Spherical harmonics not discussed in PHYS 3200

Because of the orthogonality of the Legendre polynomials, if one plugs in the relation above in the integral for the scalar magnetic potential, only the terms with $l'=1$ will survive the integration

$$\varphi_M(r,\vartheta) = \frac{M_0 R^2}{\cancel{4\pi}}\,\cancel{4\pi}\sum_{\ell' m'}\int d\cos\vartheta'\, d\phi'\, \frac{r_<^{\ell}}{r_>^{\ell+1}}\,\frac{1}{2\ell+1}\times$$

$$\times\, Y^*_{\ell' m'}(\vartheta',\phi')\,Y_{\ell' m'}(\vartheta,\phi)\,\cos\vartheta'$$

Now replace

$$\cos\vartheta' = 2\sqrt{\frac{\pi}{3}}\,Y_{10}(\cos\vartheta') = P_1(\cos\vartheta')$$

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\vartheta \; Y^*_{\ell' m'}(\vartheta,\phi)\cos\vartheta$$

$$= \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\vartheta \; Y^*_{\ell' m'}(\vartheta,\phi)\, P_1(\cos\vartheta)$$

$$= 2\sqrt{\frac{\pi}{3}} \underbrace{\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\vartheta \; Y^*_{\ell' m'}(\vartheta,\phi)\, Y_{10}(\cos\vartheta')}_{\delta_{\ell' 1}\,\delta_{m' 0}}$$

$$\varphi_M(r,\vartheta) = M_0 R^2 \frac{r_<}{r_>^2}\frac{1}{3}\underbrace{2\sqrt{\frac{\pi}{3}}\, Y_{10}(\vartheta,\phi)}_{=\cos\vartheta}$$

$$= \frac{M_0 R^2}{3}\frac{r_<}{r_>^2}\cos\vartheta$$

$\boxed{\text{Inside the sphere}}$

$$r_< = r \qquad r_> = R$$

$$\varphi_M = \frac{M_0 \cancel{R^2}}{3}\frac{r}{\cancel{R^2}}\cos\vartheta = \frac{1}{3}M_0 r\cos\vartheta = \frac{1}{3}M_0 z$$

Therefore inside the sphere one always has

$$\bar{H} = -\nabla\varphi_M = -\frac{1}{3}M_0\hat{k} = -\frac{1}{3}\bar{M}$$

$$\bar{B} = \mu_0(\bar{H}+\bar{M}) = \frac{2}{3}\mu_0\bar{M}$$

OBSERVE THAT $\bar{B} \parallel \bar{M}$, but $\bar{H}$ anti $\parallel \bar{M}$

$\boxed{\text{Outside the sphere}}$

$$r_< = R \qquad r_> = r$$

$$\varphi_M = \frac{M_0 R^2}{3}\frac{R}{r^2}\cos\vartheta = \frac{M_0 R^3}{3}\frac{\cos\vartheta}{r^2}$$

This is the potential of a magnetic dipole moment of strength

$$\bar{\mu} = \frac{4\pi}{3} R^2 \bar{M}$$

For the case of a uniformly magnetized sphere, the fields are exactly of the dipole type; for this geometry there are no contributions from higher order multiples.