Methods for solving boundary value problems in magnetostatics

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(following the approach of Jackson 5.9)

The two equations that one should start from for magnetostatic problems in a material are

$$\boxed{\nabla\cdot\bar{B} = 0} \qquad \boxed{\nabla\times\bar{H} = \bar{J}_f}$$

Vector potential method

Since $\nabla\cdot\bar{B} = 0$ then $\bar{B} = \nabla\times\bar{A}$

H is in general a complicated function of B

$$\bar{H}(\bar{B}) = \bar{H}(\nabla\times\bar{A})$$

Consequently

$$\boxed{\nabla\times\bar{H}(\nabla\times\bar{A}) = \bar{J}_f}$$

The above is in general a very complicated differential equation. However, for linear materials one

$$\bar{B} \equiv \mu\bar{H} \longrightarrow \nabla\times\left(\frac{1}{\mu}\nabla\times\bar{A}\right) = \bar{J}_f$$

If the permeability is constant over a region of space, one can conclude that

$$\nabla(\nabla\cdot\bar{A}) - \Delta\bar{A} = \mu\bar{J}_f \xrightarrow[\text{COULOMB GAUGE}]{\nabla\cdot\bar{A}=0} \boxed{\Delta\bar{A} = -\mu\bar{J}_f}$$

The solution of the Poisson like equation above should be matched across boundary surfaces using the known boundary conditions for B and H.

Magnetic scalar potential method

If the free current is zero, one can introduce a scalar potential for H. Indeed

$$\text{if} \quad J_f = 0 \longrightarrow \nabla \times \overline{H} = 0 \longrightarrow \overline{H} \equiv -\nabla \varphi_M$$

In analogy with what happens in electrostatics

$$\nabla \times \overline{E} = 0 \longrightarrow \overline{E} \equiv -\nabla \varphi$$

For the general case one can then write

$$\text{if} \quad \overline{B} = \overline{B}(\overline{H}) \longrightarrow \nabla \cdot \overline{B}(-\nabla \varphi_M) = 0$$

As usual, the relation above becomes simpler for linear media

$$\nabla \cdot (\mu \overline{H}) = -\nabla \cdot (\mu \nabla \varphi_M) \xrightarrow[\mu \text{ piecewise constant}]{} \mu \nabla \cdot (\nabla \varphi_M) = 0$$

$$\hookrightarrow \quad \Delta \varphi_M = 0 \qquad \text{LAPLACE EQUATION}$$

For a piecewise constant permeability

$$\nabla \times \overline{H} = 0 \longrightarrow \frac{1}{\mu} \nabla \times \overline{B} = 0 \longrightarrow \overline{B} \equiv -\nabla \psi_M$$

One can then write a scalar potential directly for B.

Hard ferromagnet

Hard ferromagnets are ferromagnets where the magnetization is independent from the applied field. This assumption is valid as long as the field is moderate in strength. In addition we assume to be in a case in which there are no free currents.

$$J_f = 0 \longrightarrow \nabla \times \overline{H} = 0 \longrightarrow \overline{H} = -\nabla \varphi_M \qquad \left(\underline{\text{rem}} \quad \overline{H} = \frac{\overline{B}}{\mu_0} - \overline{M}\right)$$

In addition

$$\nabla \cdot \overline{B} = \mu_0 \nabla \cdot (\overline{H} + \overline{M}) = \mu_0 \left(-\underbrace{\nabla \cdot \nabla}_{\Delta} \varphi_M + \nabla \cdot \overline{M}\right) = 0$$

$$\longrightarrow \Delta \varphi_M = \nabla \cdot \overline{M}$$

Now define

$$\rho_M \equiv -\nabla \cdot \overline{M} \longrightarrow \boxed{\Delta \varphi_M = -\rho_M}$$

MAGNETOSTATICS POISSON'S EQUATION

In analogy with the electrostatics Poisson's equation, the solution in absence of boundary surfaces is

$$\varphi_M(\overline{x}) = -\frac{1}{4\pi}\int d^3y \, \frac{\nabla_{\overline{y}} \cdot \overline{M}(\overline{y})}{|\overline{x}-\overline{y}|}$$

One can then integrate by parts the equation above by employing the identity

$$\nabla \cdot (f\overline{A}) = f(\nabla \cdot \overline{A}) + \overline{A} \cdot (\nabla f)$$

$$\varphi_M(\overline{x}) = -\frac{1}{4\pi}\int d^3y \, \nabla_{\overline{y}} \cdot \left(\frac{\overline{M}(\overline{y})}{|\overline{x}-\overline{y}|}\right)$$

$$+ \frac{1}{4\pi}\int d^3y \, \overline{M}(\overline{y}) \cdot \nabla_{\overline{y}} \frac{1}{|\overline{x}-\overline{y}|}$$

$$= -\frac{1}{4\pi}\oint_{\partial V} d\overline{S} \cdot \frac{\overline{M}(\overline{y})}{|\overline{x}-\overline{y}|} + \frac{1}{4\pi}\int d^3y \, \overline{M}(\overline{y}) \cdot \nabla_{\overline{y}} \frac{1}{|\overline{x}-\overline{y}|}$$

If the first integral vanishes (which is the case if one considers a volume larger than the magnetized material) one finds

$$\varphi_M(\overline{x}) = \frac{1}{4\pi}\int d^3y \, \overline{M}(\overline{y}) \cdot \nabla_{\overline{y}} \left(\frac{1}{|\overline{x}-\overline{y}|}\right)$$

$$= -\frac{1}{4\pi}\int d^3y \, \overline{M}(\overline{y}) \cdot \nabla_{\overline{x}} \left(\frac{1}{|\overline{x}-\overline{y}|}\right)$$

$$= -\frac{1}{4\pi} \nabla_{\overline{x}} \cdot \int d^3y \, \frac{\overline{M}(\overline{y})}{|\overline{x}-\overline{y}|}$$

Far from the region of non vanishing magnetization one can use the approximation

$$\frac{1}{|\bar{x}-\bar{y}|} \simeq \frac{1}{|\bar{x}|} \equiv \frac{1}{r}$$

$$\varphi_M(\bar{x}) = -\frac{1}{4\pi} \nabla\left(\frac{1}{r}\right) \cdot \underbrace{\int \bar{M}(\bar{y})\, d^3y}_{\bar{m}} = \frac{1}{4\pi} \frac{\bar{x}\cdot\bar{m}}{r^3}$$

magnetic dipole
moment of the material