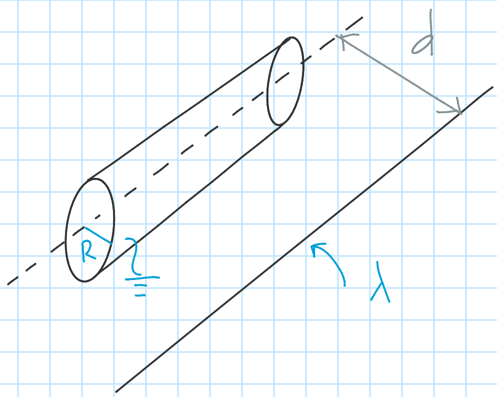
Method of images - conducting cylinder

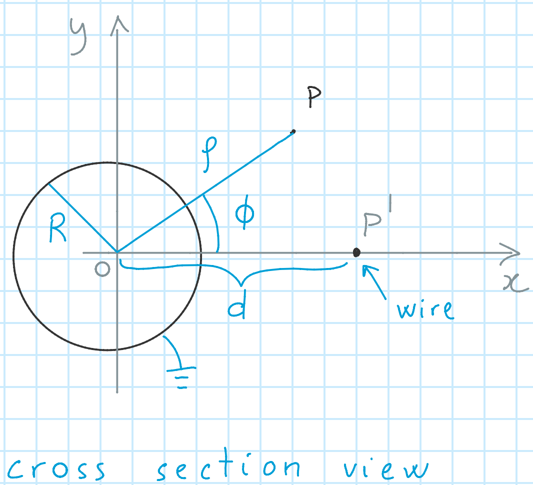
Sunday, March 3, 2019 4:01 PM

Part a) prove that the potential at a distance rho from a straight uniformly charged wire carrying charge per unit length lambda is

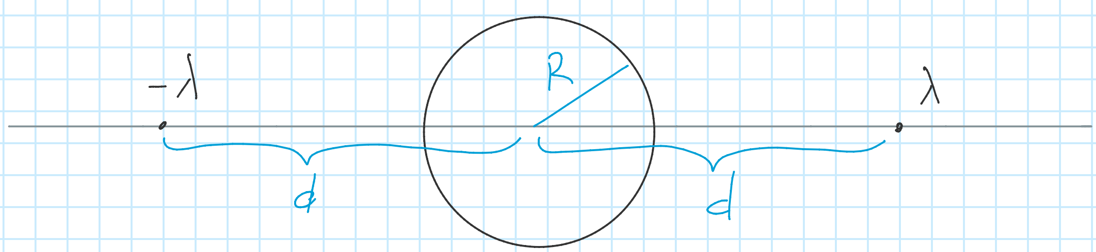
$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + \text{const}$$

Part b) use the method of images to find the potential in point P in the space surrounding an infinite conducting grounded cylinder placed parallel to a long straight uniformly charged wire. The charge density per unit length in the wire is lambda, the distance between the wire and the cylinder is d. The radius of the cylinder is R



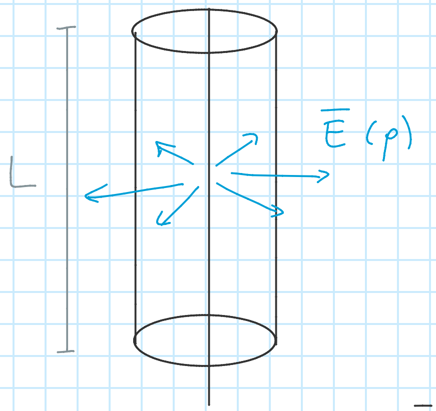


Part c) by using the result of part b, find out the potential in the case in which the infinite cylinder is isolated and is placed between two infinite straight uniformly charged wires. The two wires carry the same amount of charge per unit length, but the charges on the two wires have opposite sign.



Solution

Part a) by applying Gauss' theorem to a cylindrical gaussian surface with the axis along the wire, one can find the electric field



$$2\pi\rho L E(\rho) = \frac{\lambda L}{\varepsilon_0}$$

$$\hookrightarrow \quad E(\rho) = \frac{\lambda}{2\pi\varepsilon_0}$$

$$\varphi(\bar{r}_a) - \varphi(\bar{r}_b) = -\int_{\bar{r}_A}^{\bar{r}_B} \bar{E}\cdot d\bar{\ell} = -\int_{\rho_A}^{\rho_B} E(\rho)\, d\rho = -\int_{\rho_A}^{\rho_B} \frac{\lambda}{2\pi\varepsilon_0 \rho} d\rho$$

$$\varphi(\rho_A) - \varphi(\rho_B) = -\frac{\lambda}{2\pi\varepsilon_0} \ln\rho \Big|_{\rho_A}^{\rho_B} = -\frac{\lambda}{2\pi\varepsilon_0}\left(\ln\rho_B - \ln\rho_A\right)$$

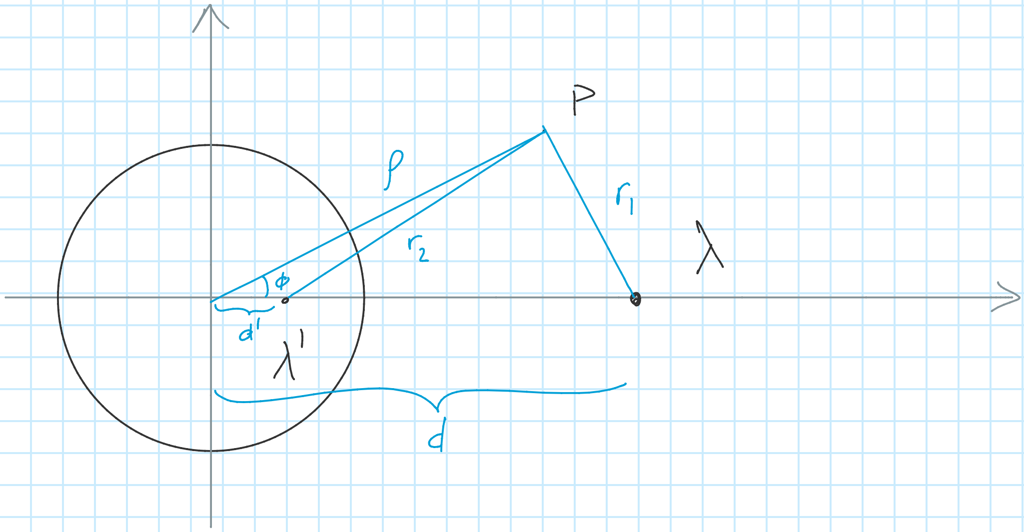
Keeping ρ_A constant one can write

$$\varphi(\rho) = -\frac{\lambda}{2\pi\varepsilon_0} \ln\rho + \text{const}$$

$$\bar{E} = -\nabla\varphi = -\frac{\partial}{\partial\rho}\varphi(\rho)\hat{\rho} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\rho}\hat{\rho} \qquad \underline{\text{q.e.d.}}$$

Part b)

Assume an image wire on the axis OP' at a distance d' from O and carrying a charge per unit length lambda'



$$\varphi = \frac{1}{2\pi\varepsilon_0}\left(-\lambda \ln r_1 - \lambda' \ln r_2 + C\right)$$

$$r_1^2 = (d - \rho\cos\varphi)^2 + \rho^2\sin^2\phi$$

$$= d^2 + \rho^2 - 2d\rho\cos\phi$$

$$r_2^2 = (\rho\cos\varphi - d')^2 + \rho^2\sin^2\phi$$

$$= (d')^2 + \rho^2 - 2\rho d'\cos\phi$$

$$r_1 = \rho\sqrt{1 + \frac{d^2}{\rho^2} - 2\frac{d}{\rho}\cos\phi}$$

$$r_2 = d'\sqrt{1 + \frac{\rho^2}{(d')^2} - 2\frac{\rho}{d'}\cos\phi}$$

if $\rho = R$ $\qquad r_1 = R\underbrace{\sqrt{1 + \frac{d^2}{R^2} - 2\frac{d}{R}\cos\phi}}_{\equiv r_3}$

$$r_2 = d' \underbrace{\sqrt{1 + \frac{\rho^2}{(d')^2} - 2\frac{\rho}{d'}\cos\phi}}_{\equiv r_4}$$

$$r_3 = r_4 \equiv r \quad \text{if} \quad \frac{d}{R} = \frac{\rho}{d'} \quad \rightarrow \quad \boxed{d' = \frac{R^2}{d}}$$

$$\varphi(\rho = R) = \frac{1}{2\pi\varepsilon_0}\left[-\lambda\left(\ln R + \ln r\right) - \lambda'\left(\ln\frac{R^2}{d} + \ln r\right) + c'\right]$$

r depends on ϕ, is not constant, but one can choose $\boxed{\lambda = -\lambda'}$

$$\varphi(\rho = R) = -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln R + \cancel{\ln r} - \ln\frac{R^2}{d} - \cancel{\ln r} - c'\right]$$

set

$$-c' = \ln\frac{R^2}{d} - \ln R = -\ln\frac{R}{d} = -\frac{1}{2}\ln\frac{R^2}{d^2}$$

$$\hookrightarrow \quad \varphi(\rho = R) = 0 \quad \text{(grounded cylinder)}$$

In a generic point P the potential is then

$$\varphi(\rho, \phi) = \frac{\lambda}{4\pi\varepsilon_0}\left[-\ln\left(d^2 + \rho^2 - 2d\rho\cos\phi\right) + \ln\left(\frac{R^4}{d^2} + \rho^2 - 2\rho\frac{R^2}{d}\cos\phi\right) - \ln\frac{R^2}{d^2}\right]$$

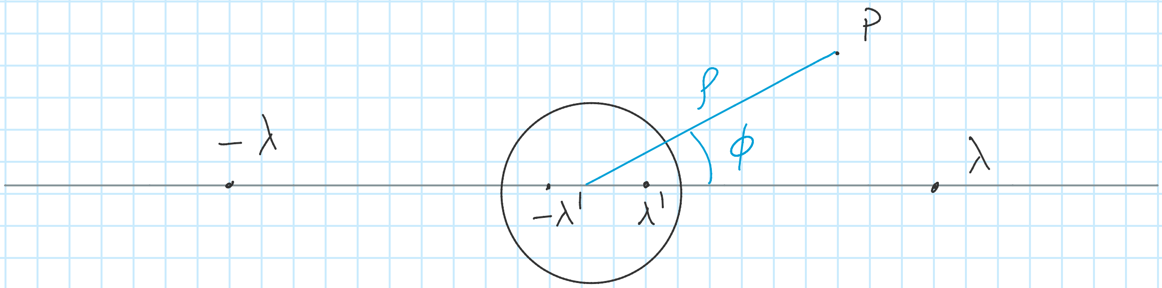
$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\left(\frac{R^4}{d^2} + \rho^2 - 2\rho\frac{R^2}{d}\cos\phi\right)\frac{d^2}{R^2}}{d^2 + \rho^2 - 2d\rho\cos\phi}\right]$$

$$\boxed{\varphi(\rho,\phi) = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{R^2+\frac{\rho^2 d^2}{R^2}-2\rho d\cos\phi}{d^2+\rho^2-2\rho d\cos\phi}}$$

Let's check that the potential is indeed zero on the surface of the cylinder

$$\varphi(R,\phi) = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{R^2+d^2-2Rd\cos\phi}{d^2+R^2-2Rd\cos\phi} = 0$$

Part c) we need to add two copies of the potential found in the previous section with image charges of the same magnitude and opposite sign. The two image charges will make the total charge on the cylinder zero. This is consistent with the fact that the cylinder is neutral and isolated.



The potential due to - lambda alone will be

$$\varphi_{-\lambda}(\rho,\phi) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{R^2+\frac{d^2\rho^2}{R^2}+2\rho d\cos\phi}{d^2+\rho^2+2d\rho\cos\phi}\right]$$

$$\boxed{\begin{aligned}\varphi(\rho,\phi) &= \varphi_\lambda(\rho,\phi)+\varphi_{-\lambda}(\rho,\phi)\\ &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{\left(R^2+\frac{d^2\rho^2}{R^2}-2\rho d\cos\phi\right)\left(d^2+\rho^2+2d\rho\cos\phi\right)}{\left(d^2+\rho^2-2d\rho\cos\phi\right)\left(R^2+\frac{d^2\rho^2}{R^2}+2d\rho\cos\phi\right)}\right]\end{aligned}}$$