Magnetization

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In presence of a magnetic field, matter becomes magnetized. Microscopically matter behaves as a collection of tiny dipoles, which will tend to align in one direction when exposed to a magnetic field.

No matter what is the cause of magnetization, the state of magnetic polarization can be described by the magnetization vector M.

$\bar{M}$ = magnetic dipole moment per unit volume

M could be due to paramagnetism, diamagnetism, or ferromagnetism; we will take M as given and calculate what is the contribution to the total magnetic field due to the magnetization of the material.

Field of a magnetized object

Vector potential of a single magnetic dipole

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \frac{\bar{m} \times \hat{\imath}}{\imath^2}$$

rem $\bar{\imath} = \bar{x} - \bar{y}$

$\imath = |\bar{x} - \bar{y}|$

where $\bar{y}$ is the position of $\bar{m}$

So the magnetization M gives the following contribution to the vector potential

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int d^3y \, \frac{\bar{M}(\bar{y}) \times (\bar{x} - \bar{y})}{|\bar{x} - \bar{y}|^3} = \frac{\mu_0}{4\pi} \int d^3y \, \frac{\bar{M}(\bar{y}) \times \hat{\imath}}{\imath^2}$$

One then can apply the identity

$$\nabla_{\bar{y}} \frac{1}{\imath} = \frac{\hat{\imath}}{\imath^2}$$

$$\partial_{y_i} \frac{1}{\imath} = -\frac{1}{\imath^2} \frac{1}{2} \, 2 (x_i - y_i)(-1) = \frac{x_i - y_i}{\imath^3} = \frac{\hat{\imath}_i}{\imath^2}$$

$$\partial_{y_i} \frac{1}{r} = -\frac{1}{r^2}\,\frac{1}{2}\,2\,(x_i - y_i)(-1) = \frac{x_i - y_i}{r^3} = \frac{r_i}{r^2}$$

One can then rewrite the vector potential

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \int \bar{M}(\bar{y}) \times \left(\nabla_{\bar{y}} \frac{1}{r} \right) d^3y$$

One can then use the identity

$$\nabla \times (f \bar{A}) = f\,(\nabla \times \bar{A}) - \bar{A} \times (\nabla f)$$

With

$$f \to \frac{1}{r} \qquad\qquad \bar{A} \to \bar{M}$$

One can then write

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi} \left\{ \int \frac{1}{r} \left(\nabla_{\bar{y}} \times \bar{M}(\bar{y}) \right) d^3y - \int \nabla_{\bar{y}} \times \left(\frac{\bar{M}(\bar{y})}{r} \right) d^3y \right\}$$

Now we use the following theorem for the integral of a curl over a volume

$$\boxed{\int_V (\nabla \times \bar{F})\, d^3x = -\int_{\partial V} \bar{F} \times d\bar{s}}$$

Proof: Write the curl in component

$$(\nabla \times \bar{F})_z = \partial_x F_y - \partial_y F_x = \nabla \cdot \underbrace{(F_y, -F_x, 0)}_{\bar{G}} = \nabla \cdot \bar{G}$$

$$\int_V (\nabla \times \bar{F})_z\, d^3x = \int_V \nabla \cdot \bar{G}\, d^3x = \int_{\partial V} \bar{G} \cdot d\bar{s}$$

One can then observe that

$$\bar{G} \cdot d\bar{s} = F_y\, dS_x - F_x\, dS_y = -(\bar{F} \times d\bar{s})_z$$

Therefore

$$\int_V (\nabla \times \bar{F})_z\, d^3x = -\int_{\partial V} (\bar{F} \times d\bar{s})_z$$

This can be repeated for all components, that proves the theorem.

Consequently, one has

$$\bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int_V \frac{1}{r}\left[\nabla_y \times \bar{M}(\bar{y})\right] d^3y + \frac{\mu_0}{4\pi}\oint_{\partial V}\frac{1}{r}\left[\bar{M}(\bar{y})\times d\bar{s}\right]$$

The first term looks like the potential of a volume current

$$\bar{J}_b \equiv \nabla \times \bar{M} \qquad \left(\text{rem} \quad \bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int \frac{\bar{J}}{r}\, d^3x\right)$$

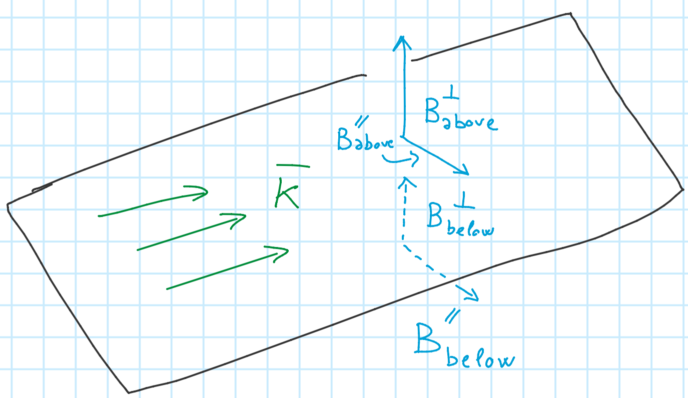
The second term is the potential due to a surface current

$$\bar{K}_b = \bar{M}\times\hat{n} \qquad \left(\text{rem} \quad \bar{A}(\bar{x}) = \frac{\mu_0}{4\pi}\int \frac{\bar{K}\, ds}{r}\right)$$

Notice the similarity with the case of polarization due to the electric field

$$\rho_b = -\nabla\cdot\bar{P} \qquad \text{and} \qquad \sigma_b = \bar{P}\cdot\hat{n}$$

Remember what are the boundary conditions for the magnetic field across a surface carrying a surface current density



$$B^{\perp}_{above} = B^{\perp}_{below} \qquad\qquad B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

The two conditions above can be written in a single equation in a compact way

$$\bar{B}_{above} - \bar{B}_{below} = \mu_0\left(\bar{K}\times\hat{n}\right)$$

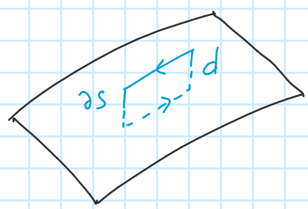
Notice that if we work in the Coulomb's gauge the vector potential A is continuous across the surface. One can indeed observe that

$$\nabla \cdot \bar{A} = 0 \quad , \quad \bar{B} = \nabla \times \bar{A}$$

$$\oint_{\partial S} \bar{A} \cdot d\bar{\ell} = \int_S (\nabla \times \bar{A}) \cdot d\bar{s} = \int_S \bar{B} \cdot d\bar{s} = \Phi_B$$

magnetic flux

now choose



$d \to 0 \Rightarrow \oint \bar{A} \cdot d\bar{\ell} = 0$

If one sends d to zero the area in the loop goes to zero and the magnetic flux through that area goes to zero. Therefore the component of the vector potential parallel to the surface is continuous.

The normal derivative of the vector potential is discontinuous across the surface. In fact, if one remembers that

$$\nabla(\underbrace{\hat{n} \cdot \bar{A}}) = \hat{n} \times (\nabla \times \bar{A}) + \bar{A} \times \cancelto{0}{(\nabla \times \hat{n})} + (\hat{n} \cdot \nabla)\bar{A} + \cancelto{0}{(\bar{A} \cdot \nabla)\hat{n}}$$

$\bar{A}$ cannot have a component // to $\hat{n}$

$$\hat{n} \times \bar{B} = \hat{n} \times (\nabla \times \bar{A}) = -(\hat{n} \cdot \nabla)\bar{A}$$

Consequently

$$\hat{n} \times \bar{B}_{above} - \hat{n} \times \bar{B}_{below} = \mu_0 \overbrace{\hat{n} \times (\bar{k} \times \hat{n})}^{-\bar{k}}$$

$$-\hat{n} \cdot \nabla \bar{A}_{above} + \hat{n} \cdot \nabla \bar{A}_{below} = -\mu_0 \bar{k}$$

$$\frac{\partial}{\partial n} \bar{A}_{below} - \frac{\partial}{\partial n} \bar{A}_{above} = -\mu_0 \bar{k}$$

Compare the boundary conditions for E and B at a surface

$$\bar{E}_{above} - \bar{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n} \qquad \bar{B}_{above} - \bar{B}_{below} = \mu_0 (\bar{K} \times \hat{n})$$