Legendre polynomials and linear dielectrics

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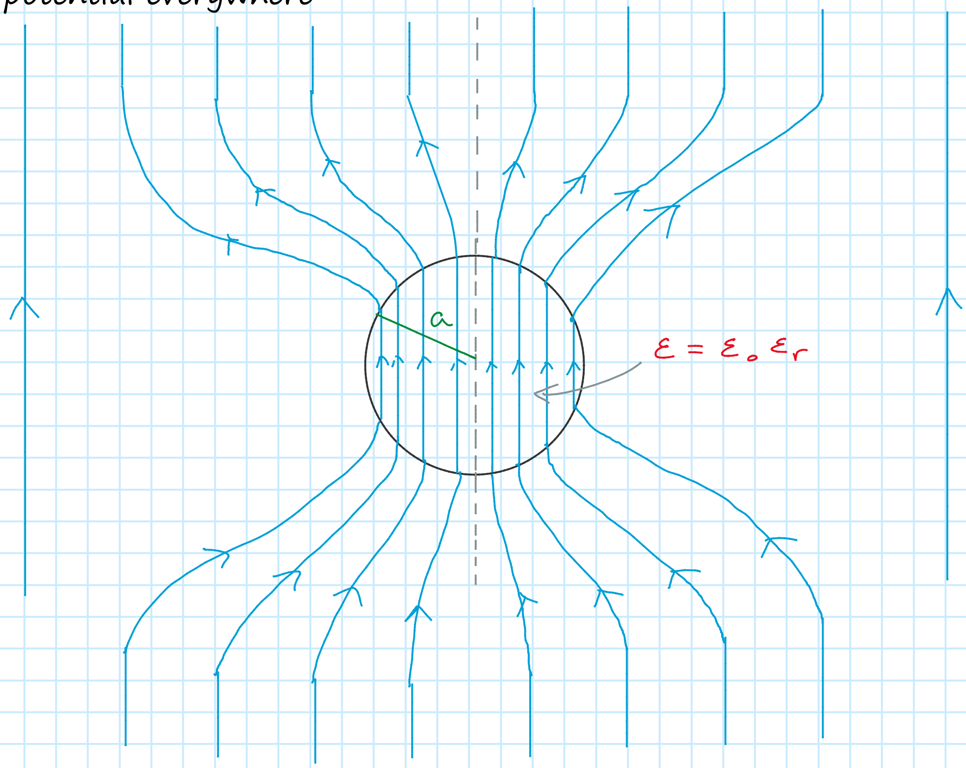
A sphere of homogenous linear dielectric is placed in an otherwise uniform field E_0. Our goal is to find the potential everywhere



We need to impose the following boundary conditions:

1) The potential is continuous in r = a

$$\lim_{r \to a^-} \varphi_{in} = \lim_{r \to a^+} \varphi_{out} \qquad (i)$$

2) The component of the electric displacement perpendicular to the surface of the sphere is continuous

$$D_{in}^{\perp} = D_{out}^{\perp} \qquad \varepsilon \left.\frac{\partial \varphi_{in}}{\partial r}\right|_{r=a} = \varepsilon_0 \left.\frac{\partial \varphi_{out}}{\partial r}\right|_{r=a} \qquad (ii)$$

3) Very far away from the dielectric sphere one should have

$$\boxed{\text{for} \quad r \gg a} \qquad \varphi = -E_0 z = -E_0 \underbrace{r\cos\theta}_{z}$$

$$\bar{E} = -\nabla\varphi = E_0 \hat{k}$$

The problem has azimuthal symmetry. One can expand the potential in Legendre polynomials

$$\varphi_{in}(r,\vartheta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\vartheta)$$

since $\varphi(r=0)$ must be finite

$$\varphi_{out}(r,\vartheta) = -E_0 r\cos\vartheta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\vartheta)$$

From the condition (i) one finds

for $\ell = 1$

$$\boxed{A_1 a = -E_0 a + \frac{B_1}{a^2}} \qquad (a)$$

for $\ell \neq 1$

$$\boxed{A_\ell a^\ell = \frac{B_\ell}{a^{\ell+1}}} \qquad (b)$$

From the condition (ii) one finds

$$\varepsilon \sum_{\ell=0}^{\infty} A_\ell \, \ell \, a^{\ell-1} P_\ell(\cos\vartheta) = -\varepsilon_0 E_0 \cos\vartheta - \sum_{\ell=0}^{\infty} \varepsilon_0 (\ell+1) \frac{B_\ell}{a^{\ell+2}} P_\ell(\cos\vartheta)$$

for $\ell = 1$

$$\boxed{\varepsilon A_1 = -\varepsilon_0 E_0 - 2\varepsilon_0 \frac{B_1}{a^3}} \qquad (c)$$

for $\ell \neq 1$

$$\boxed{\varepsilon \ell A_\ell a^{\ell-1} = -\varepsilon_0 \frac{(\ell+1)}{a^{\ell+2}} B_\ell} \qquad (d)$$

From (b) one finds $\quad B_\ell = A_\ell a^{2\ell+1}$

from (d) $\quad \varepsilon \ell A_\ell a^{\ell-1} = -\varepsilon_0 \frac{\ell+1}{a^{\ell+2}} A_\ell a^{2\ell+1}$

$$\boxed{A_\ell = 0 \quad \forall \ell \neq 1}$$

from (a)

$$B_1 = a^2 (A_1 a + E_0 a) = a^3 (A_1 + E_0)$$

from (c)

$$\varepsilon A_1 = -\varepsilon_0 E_0 - 2\varepsilon_0 \frac{1}{\cancel{a^3}} \underbrace{\cancel{a^3}(A_1 + E_0)}_{B_1}$$

$$\varepsilon A_1 = -\varepsilon_0 E_0 - 2\varepsilon_0 A_1 - 2\varepsilon_0 E_0$$

$$A_1 (\varepsilon + 2\varepsilon_0) = -3\varepsilon_0 E_0$$

$$\boxed{A_1 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 = -\frac{3}{\frac{\varepsilon}{\varepsilon_0} + 2} E_0}$$

$$B_1 = a^3 \left(-\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 + E_0 \right)$$

$$= a^3 E_0 \left(\frac{-3\varepsilon_0 + \varepsilon + 2\varepsilon_0}{\varepsilon + 2\varepsilon_0} \right)$$

$$= \boxed{a^3 E_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \right) = a^3 E_0 \frac{\frac{\varepsilon}{\varepsilon_0} - 1}{\frac{\varepsilon}{\varepsilon_0} + 2}}$$

One can then finally write the potential

$$\boxed{\varphi_{in} = -\frac{3}{\frac{\varepsilon}{\varepsilon_0}+2}\, r E_0 \cos\vartheta}$$

$r < a$

$$\boxed{\varphi_{out} = -E_0 r \cos\vartheta + \frac{E_0 a^3}{r^2}\,\frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+1}\cos\vartheta}$$

$r > a$

The field inside the sphere is uniform and directed along z

$$\bar{E} = -\nabla\varphi = +\frac{3}{\frac{\varepsilon}{\varepsilon_0}+2} E_0\left(\frac{\partial}{\partial z} r\cos\vartheta\right)\hat{k}$$

$$= +\frac{3}{\frac{\varepsilon}{\varepsilon_0}+2} E_0\,\hat{k}$$

The potential outside the sphere can be interpreted as the superposition of a constant field along z and a dipole. Indeed remember that

$$\varphi_{dipole} = \frac{1}{4\pi\varepsilon_0}\frac{\bar{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\vartheta}{r^2}$$

In our case

$$p = E_0 a^3 \frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}$$

The polarization per unit volume inside the sphere is

$$\bar{P} = \varepsilon_0 \chi_e \bar{E} = \varepsilon_0\left(\frac{\varepsilon}{\varepsilon_0}-1\right)\bar{E}$$

$$= (\varepsilon-\varepsilon_0)\frac{3}{\frac{\varepsilon}{\varepsilon_0}+2}\bar{E}_0 = 3\varepsilon_0\left(\frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}\right)\bar{E}_0$$

$$\boxed{\bar{P} = 3\varepsilon_0\frac{\frac{\varepsilon}{\varepsilon_0}-1}{\frac{\varepsilon}{\varepsilon_0}+2}\bar{E}_0}$$

The bound charge on the surface becomes then

$$\sigma_b = \bar{P} \cdot \hat{n} = \bar{P} \cdot \hat{r} = 3\varepsilon_0 \left(\frac{\frac{\varepsilon}{\varepsilon_0} - 1}{\frac{\varepsilon}{\varepsilon_0} + 1} \right) E_0 \cos\theta$$

The polarization inside the sphere lowers the value of the electric field inside the sphere, since

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} > 1 \quad \Rightarrow \quad \frac{3}{\frac{\varepsilon}{\varepsilon_0} + 2} < 1$$