Method of images in a problem within a dielectric

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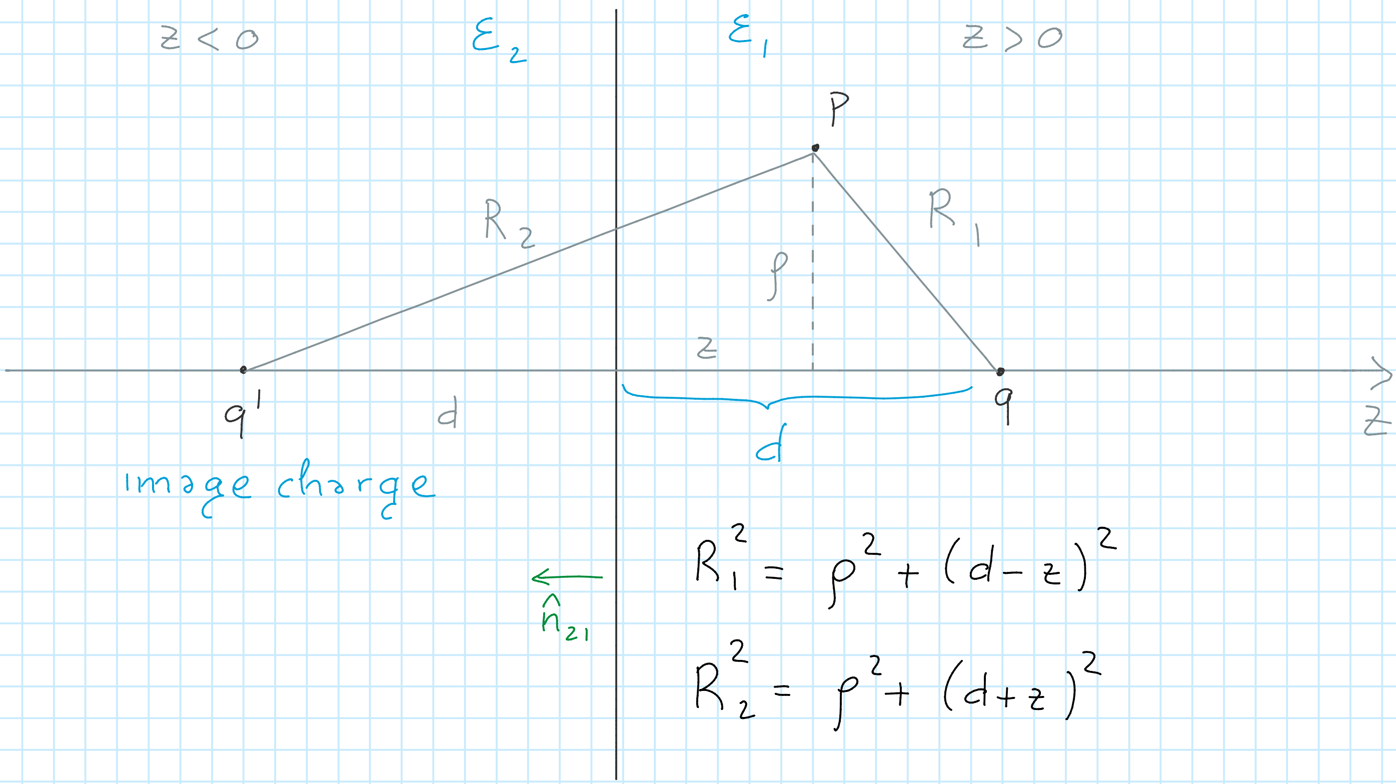
The following problem, including a point charge in a dielectric and an interface between two different dielectrics, can be solved with the method of images (this physical situation is discussed in Jackson, section 4.4).



$$R_1^2 = \rho^2 + (d-z)^2$$

$$R_2^2 = \rho^2 + (d+z)^2$$

The fields satisfy the following equations

$$z>0 \qquad \nabla\cdot\bar{D}_1 = \rho_f \;\longrightarrow\; \varepsilon_1 \nabla\cdot\bar{E} = \rho_f$$

$$z<0 \qquad \nabla\cdot\bar{D}_2 = 0 \;\longrightarrow\; \varepsilon_2 \nabla\cdot\bar{E} = 0$$

$$\nabla\times\bar{E} = 0 \qquad \text{everywhere}$$

The conditions at the boundary are

$$\lim_{z\to 0^-}\begin{Bmatrix} E_x \\ E_y \end{Bmatrix} - \lim_{z\to 0^+}\begin{Bmatrix} E_x \\ E_y \end{Bmatrix} = 0 \qquad \text{continuity of } \bar{E}_{\parallel}$$

$$\bar{D}^{\perp}_{above} - \bar{D}^{\perp}_{below} = \sigma_{free} = 0$$

$$\lim_{z \to 0^-} \varepsilon_2 E_z = \lim_{z \to 0^+} \varepsilon_1 E_z$$

Now let's use the following Ansatz for the potential in the region $z > 0$

$$\boxed{\varphi_R(\rho, \phi, z) = \frac{1}{4\pi \varepsilon_1} \left(\frac{q}{R_1} + \frac{q'}{R_2} \right)}$$

$z > 0$

Since in the region $z < 0$ there are no free charges, the potential is the solution of Laplace's equation. The simplest ansatz for this region is the potential due to a second image charge q'' placed where the physical charge q is

$$\boxed{\varphi_L(\rho, z, \phi) = \frac{1}{4\pi \varepsilon_2} \frac{q''}{R_1}}$$

$z < 0$

In order to fix q' and q'' one needs to enforce the boundary conditions for the fields

$$\frac{\partial}{\partial z} \frac{1}{R_1} = \frac{\partial}{\partial z} \frac{1}{\sqrt{\rho^2 + (d-z)^2}} = +\frac{1}{2} \frac{1}{R_1^3} 2(d-z)$$

$$= \frac{d-z}{\left(\sqrt{\rho^2 + (d-z)^2}\right)^3} \longrightarrow \boxed{\left.\frac{\partial}{\partial z} \frac{1}{R_1}\right|_{z=0} = \frac{d}{(\rho^2 + d^2)^{\frac{3}{2}}}}$$

$$\frac{\partial}{\partial z} \frac{1}{R_2} = -\frac{1}{2} \frac{1}{R_2^3} 2(d+z) = -\frac{d+z}{\left(\sqrt{\rho^2 + (d+z)^2}\right)^3}$$

$$\boxed{\frac{\partial}{\partial z}\frac{1}{R_2}\bigg|_{z=0} = -\frac{d}{(\rho^2+d^2)^{\frac{3}{2}}}}$$

Consequently

$$\lim_{z\to 0^+} \varepsilon_1 E_z = \lim_{z\to 0^+} \varepsilon_1\left(-\frac{\partial \varphi_R}{\partial z}\right) =$$

$$= \frac{1}{4\pi}\left(-\frac{qd}{(\rho^2+d^2)^{\frac{3}{2}}} + \frac{q'd}{(\rho^2+d^2)^{\frac{3}{2}}}\right)$$

$$= \frac{1}{4\pi}\frac{d}{(\rho^2+d^2)^{\frac{3}{2}}}(q'-q)$$

$$\lim_{z\to 0^-} \varepsilon_2 E_z = \lim_{z\to 0^-} \varepsilon_2\left(-\frac{\partial \varphi_L}{\partial z}\right) = \frac{1}{4\pi}\frac{-q''d}{(\rho^2+d^2)^{\frac{3}{2}}}$$

$$\lim_{z\to 0^-} \varepsilon_2 E_z = \lim_{z\to 0^+} \varepsilon_1 E_z \longrightarrow \boxed{q'' = q - q'}$$

From the boundary condition for the field component on the x-y plane one finds

$$\lim_{z\to 0^+} E_\rho = \lim_{z\to 0^+}\left(-\frac{\partial \varphi_R}{\partial \rho}\right)$$

$$= \lim_{z\to 0^+}\frac{-1}{4\pi\varepsilon_1}\left(q\frac{\partial}{\partial\rho}\frac{1}{R_1} + q'\frac{\partial}{\partial\rho}\frac{1}{R_2}\right)$$

$$= \lim_{z\to 0^+} -\left(-\frac{1}{\cancel{2}}\frac{q}{R_1^3}\cancel{2}\rho - q'\frac{\rho}{R_2^3}\right) = \frac{1}{4\pi\varepsilon_1}\frac{\rho}{(\rho^2+d^2)^{\frac{3}{2}}}(q+q')$$

$$\lim_{z\to 0^-} E_\rho = -\frac{1}{4\pi\varepsilon_2}\left(-\frac{1}{2}\,\frac{q'' z \rho}{R_1^3}\right) = \frac{1}{4\pi\varepsilon_2}\,\frac{\rho q''}{(\rho^2+d^2)^{\frac{3}{2}}}$$

$$\lim_{z\to 0^-} E_\rho = \lim_{z\to 0^+} E_\rho \longrightarrow \boxed{\frac{q+q'}{\varepsilon_1} = \frac{q''}{\varepsilon_2}}$$

One can then solve the two frames equations w.r.t. q' and q''

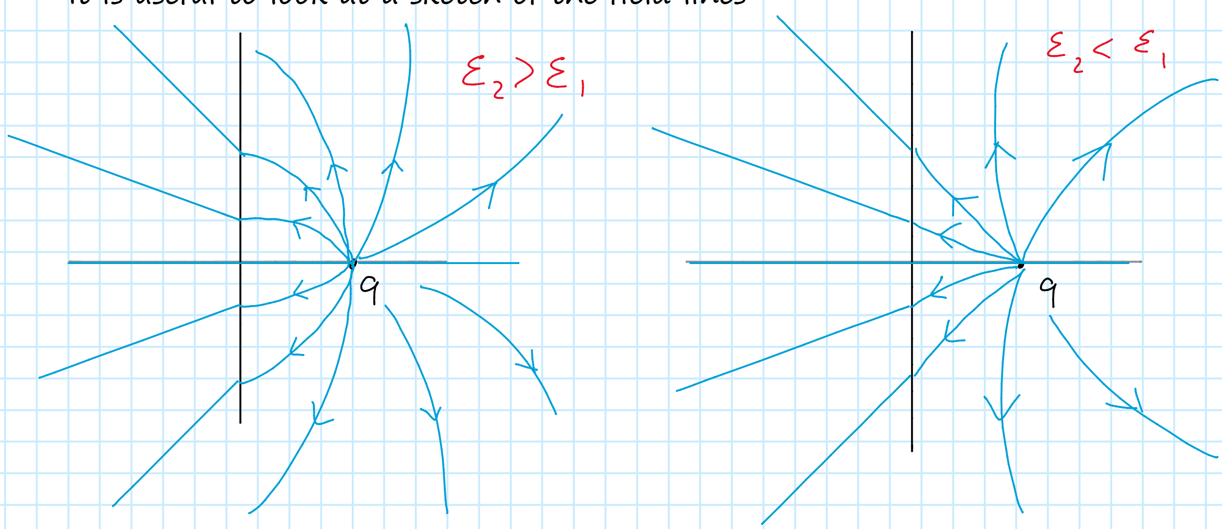
$$\frac{1}{\varepsilon_1}(q+q') = \frac{1}{\varepsilon_2}(q-q')$$

$$q\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right) = -q'\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)$$

$$q\,\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2\varepsilon_1} = -q'\,\frac{\varepsilon_2+\varepsilon_1}{\varepsilon_2\varepsilon_1} \longrightarrow \boxed{q' = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2+\varepsilon_1}\,q}$$

$$q'' = q - q' = \left(1 - \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_2+\varepsilon_1}\right)q \qquad \boxed{q'' = \frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}\,q}$$

It is useful to look at a sketch of the field lines



Inside each one of the dielectrics

$$\left.\begin{aligned}\nabla\cdot\bar{D} &= 0\\ \bar{D} &= \varepsilon_i\bar{E}\end{aligned}\right\} \rightarrow \nabla\cdot\bar{E} = 0$$

One can then calculate the polarization of the dielectric and the bound charge density

$$\overline{P} = \varepsilon_0 \chi_e \overline{E} \longrightarrow \rho_b = -\nabla\cdot\overline{P} = -\varepsilon_0\chi_e \nabla\cdot\overline{E} = 0$$

The above applies everywhere except at the location of the physical charge q.

The surface charge density at the interface between the two dielectrics can be calculated as the sum of the bound surface charge density due to the polarization in each one of the two regions

$$\sigma = +\overline{P}_2\cdot\hat{n}_{12} + \overline{P}_1\cdot\hat{n}_{21} = -\left(\overline{P}_2 - \overline{P}_1\right)\cdot\hat{n}_{21}$$

unit vector pointing out of the region with permittivity ε_2

$\hat{n}_{12} = -\hat{n}_{21}$

polarization in region of permittivity ε_i

In addition one knows that

$$\overline{D}_i = \varepsilon_0\overline{E} + \overline{P}_i = \varepsilon_i\overline{E} \longrightarrow \overline{P}_i = (\varepsilon_i - \varepsilon_0)\overline{E}$$

$$\overline{P}_i = -(\varepsilon_i - \varepsilon_0)\nabla\varphi$$

rem direction of $\hat{n}_{21}$

$$\overline{P}_i\cdot\hat{n}_{21} = -(\varepsilon_i - \varepsilon_0)\nabla\varphi\cdot\hat{n}_{21} = -(\varepsilon_i - \varepsilon_0)E_z\Big|_{z=0}$$

Therefore

$$\sigma = -\overline{P}_2\cdot\hat{n}_{21} + \overline{P}_1\cdot\hat{n}_{21}$$

$$= (\varepsilon_2 - \varepsilon_0)E_z\Big|_{z\to 0^-} - (\varepsilon_1 - \varepsilon_0)E_z\Big|_{z\to 0^+}$$

$$= (\varepsilon_2 - \varepsilon_0)\frac{1}{4\pi\varepsilon_2}\left(-\frac{q''d}{(\rho^2+d^2)^{\frac{3}{2}}}\right)$$

$$+ (\varepsilon_1 - \varepsilon_0)\frac{1}{4\pi\varepsilon_1}\left[-\frac{(q'-q)d}{(\rho^2+d^2)^{\frac{3}{2}}}\right]$$

$$= -\frac{1}{4\pi}\frac{d}{(\rho^2+d^2)^{\frac{3}{2}}}\left[\frac{\varepsilon_2-\varepsilon_0}{\varepsilon_2}q'' + \frac{\varepsilon_1-\varepsilon_0}{\varepsilon_1}(q'-q)\right]$$

One can then plug the expressions of the image charges as a function of q in the square bracket above

$$q'' = \frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}q \qquad\qquad q' = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$$

$$\left[\ldots\right] = \frac{\varepsilon_2-\varepsilon_0}{\cancel{\varepsilon_2}}\frac{2\cancel{\varepsilon_2}}{\varepsilon_1+\varepsilon_2}q + \frac{\varepsilon_1-\varepsilon_0}{\varepsilon_1}\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}-1\right)q$$

$$= \frac{\varepsilon_2-\varepsilon_0}{\varepsilon_1+\varepsilon_2}2q + \frac{\varepsilon_1-\varepsilon_0}{\varepsilon_1}\frac{\cancel{\varepsilon_1}-\varepsilon_2-\cancel{\varepsilon_1}-\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$$

$$= \frac{q}{\varepsilon_1+\varepsilon_2}\left[2(\varepsilon_2-\varepsilon_0) - 2\frac{\varepsilon_1-\varepsilon_0}{\varepsilon_1}\varepsilon_2\right]$$

$$= \frac{2q}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}\left[\cancel{\varepsilon_1\varepsilon_2} - \varepsilon_1\varepsilon_0 - \cancel{\varepsilon_1\varepsilon_2} + \varepsilon_0\varepsilon_2\right]$$

$$= \frac{2q\varepsilon_0}{\varepsilon_1(\varepsilon_1+\varepsilon_2)}(\varepsilon_2-\varepsilon_1)$$

One can finally write the bound surface charge density as

$$\sigma = -\frac{1}{2\pi}\frac{qd}{(\rho^2+d^2)^{\frac{3}{2}}}\frac{\varepsilon_0}{\varepsilon_1}\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)$$