The electric displacement D

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We found that the polarization of the dielectric produces bound volume and surface charge densities which are related to the polarization vector P:

$$\rho_b = -\nabla \cdot \bar{P} \qquad \sigma_b = \bar{P} \cdot \hat{n}$$

Now let's assume that the total charge density if the sum of a free charge density, which we assume to control experimentally, and of the bound charge density.

$$\rho = \rho_f + \rho_b$$

One then applies Gauss' law for the electric field

$$\varepsilon_0 \nabla \cdot \bar{E} = \rho = \rho_f + \rho_b = -\nabla \cdot \bar{P} + \rho_f$$

The electric field in the equation above is the total field, due to both free and bound charges. One can then rewrite the relation above as

$$\nabla \cdot (\underbrace{\varepsilon_0 \bar{E} + \bar{P}}_{\equiv \text{ ELECTRIC DISPLACEMENT } \bar{D}}) = \rho_f$$

$$\boxed{\nabla \cdot \bar{D} = \rho_f \quad , \quad \bar{D} \equiv \varepsilon_0 \bar{E} + \bar{P}}$$

The differential equation satisfied by the electric displacement can be written in integral form as

$$\oint_{\partial V} \bar{D} \cdot d\bar{s} = Q_f$$

free charge enclosed in the volume V

Notice that we did not consider the bound surface charge density in deriving the differential equation for D. We cannot apply the equation for D exactly at the surface, because the divergence of D includes a piece that comes from the divergence of the polarization P. Since the polarization drops suddenly to zero going from the inside to the outside of the material, its divergence is a delta function, and therefore the l.h.s. of the equation we found blows up. A more realistic model of what happens near the edge would be to have a bound volume charge density that varies rapidly near the edge of the material but it is still a smooth continuous function of the coordinate perpendicular to the surface. In that picture one can apply the differential form of the equation for D everywhere.

Warning

We saw that

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \longrightarrow \bar{E}(\bar{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\bar{y})(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^3} d^3y$$

Can one then conclude that

$$\nabla \cdot \bar{D} = \rho_f \overset{?}{\longrightarrow} \bar{D}(\bar{x}) = \frac{1}{4\pi} \int \frac{\rho(\bar{y})\ (\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^3} d^3y$$

One needs to know the divergence and the curl of a vector in order to fully determine it. For the field E one has

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \qquad \underline{\text{and}} \qquad \nabla \times \bar{E} = 0$$

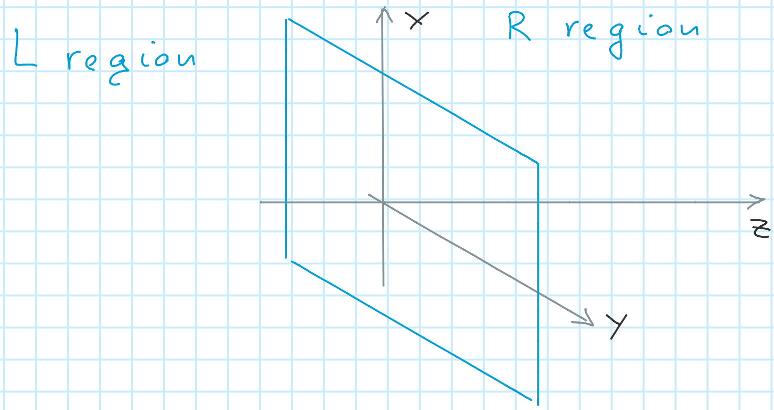
It is the curl equation that allows one to conclude that the electric field E can be written as the gradient of a scalar potential and then to arrive to the integral form written above. But the curl of the electric displacement D is not zero in general

$$\nabla \times \bar{D} = \nabla \times \left(\varepsilon_0 \bar{E} + \bar{P}\right) = \nabla \times \bar{P} \neq 0$$

There is no potential for D, and D is not fixed completely by the free charge density per unit volume.

Electric displacement across a boundary surface

In analogy with what was done for the electric field, one can find out which components of the electric displacement are continuous and which are discontinuous across a boundary surface.



$$\overline{D} = \overline{D}_L \vartheta(-z) + \overline{D}_R \vartheta(+z)$$

$$\rho_f = \rho_{f,L} \vartheta(-z) + \rho_{f,R} \vartheta(+z) + \sigma_f(x,y)\,\delta(z)$$

$$\nabla\cdot\overline{D} = \nabla\cdot\overline{D}_L \vartheta(-z) - \overline{D}_L\cdot\hat{k}\,\delta(z) + \nabla\cdot\overline{D}_R \vartheta(z) + \overline{D}_R\cdot\hat{k}\,\delta(z)$$

$$\nabla\cdot\overline{D} = \rho_f \longrightarrow \boxed{\left[\left(\overline{D}_R - \overline{D}_L\right)\cdot\hat{k}\right]_{z=0} = \sigma_f}$$

Remember that we already know that

$$\left[\left(\overline{E}_R - \overline{E}_L\right)\cdot\hat{k}\right]_{z=0} = \frac{\sigma}{\varepsilon}$$

One can analyze in the same way the relation

$$\nabla\times\bar{D} = \nabla\times\bar{P}$$

$$\nabla\times\bar{D} = \vartheta(-z)\,\nabla\times\bar{D}_L + \vartheta(+z)\,\nabla\times\bar{D}_R + \hat{k}\times\left(\bar{D}_R-\bar{D}_L\right)\delta(z)$$

$$\nabla\times\bar{P} = \vartheta(-z)\,\nabla\times\bar{P}_L + \vartheta(+z)\,\nabla\times\bar{P}_R + \hat{k}\times\left(\bar{P}_R-\bar{P}_L\right)\delta(z)$$

$$\boxed{\left[\hat{k}\times\left(\bar{D}_R-\bar{D}_L\right)\right]_{z=0} = \left[\hat{k}\times\left(\bar{P}_R-\bar{P}_L\right)\right]_{z=0}}$$

Remember that we already know that the component of E parallel to the surface is continuous

$$\left[\hat{k}\times\left(\bar{E}_R-\bar{E}_L\right)\right]_{z=0} = 0$$